

Living With Lambda

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This is a short pedagogical introduction to the consequences of a nonzero cosmological constant Λ in physical cosmology.

The cosmological constant is an energy associated with the vacuum, that is, with “empty space”. The possibility of a nonzero cosmological constant Λ has been entertained several times in the past for theoretical and observational reasons (early work includes e.g. Einstein 1917, Petrosian, Salpeter & Szekeres 1967, Gunn & Tinsley 1975, a popularized description of the history is found in Goldsmith 1997). Recent supernovae results (Perlmutter et al 1998, Riess et al 1998) have made a strong case for a nonzero and possibly quite large cosmological constant. Their results have encouraged increased interest in the properties of a universe with nonzero cosmological constant. Several other observations of various cosmological phenomena are also planned or underway which will further constrain the range of allowed values for the cosmological constant. Given the expected quality and quantity of upcoming data, there is reason to believe that we will know soon whether or not we need to learn to “live with lambda”.

The purpose of this review is to provide a short pedagogical introduction to the consequences of a nonzero cosmological constant. Basic terms are defined in section one, where the equations for the time evolution of the scale factor of the universe (defined below) are given. Section two indicates the current theoretically expected values of the cosmological constant, introducing the theoretical “cosmological constant problem.” Some suggestions to explain a cosmological constant consistent with current measurements are listed. In section three, the time evolution of the scale factor of the universe (from section one) is used to show how the age of the universe, the path length travelled by light, and other properties depending on the spacetime geometry vary when the cosmological constant is present. Section four outlines some effects of a nonzero cosmological constant on structure formation. Section five summarizes how some recent and upcoming measurements may constrain Λ . Several observations (including the supernovae) are described which have provided constraints or show promise for the future. These observations are improving rapidly. The current theoretical explanations for a nonzero cosmological constant consistent with the data, some of which are listed in section two, are not compelling. Section six briefly describes some suggested theoretical alternatives to a nonzero cosmological constant. Section seven contains a description of the future of the universe if the cosmological constant is nonzero and then summarizes.

Earlier reviews, in particular the one by Carroll, Press and Turner (1992, hereafter denoted by CPT) are highly recommended for some of the in depth results and more references, as well as the books by Kolb & Turner (1990), Peebles (1993) and Padmanabhan (1993) for the basic cosmology. The referencing is indicative rather than comprehensive, for more extensive referencing consult the more in depth reviews, textbooks and articles cited.

I. INTRODUCTION

The cosmological constant is a constant energy density associated with “empty space.” Its presence affects the properties of spacetime (specified by a metric) and matter (stress energy). The metric of spacetime and the stress energy tensor of matter are related via Einstein’s equations.

Thus a first step in studying the effects of the cosmological constant is to specify the metric and stress energy tensor and then to relate them via Einstein’s equations. For a spatially homogeneous spacetime the metric can be written as

$$ds^2 = c^2 dt^2 - R^2(t) \{ d\chi^2 + r^2(\chi) [d\theta^2 + \sin^2 \theta d\phi^2] \} . \quad (1)$$

In the above, the expansion or contraction of space with time is given by $R(t)$, and $d\chi = dr / (1 - kr^2)^{1/2}$, with $k = (-1, 0, 1)$ for an open, flat, or closed universe respectively. Thus, using the notation of CPT,

$$r(\chi) = \text{sinn } \chi = \begin{cases} \sin \chi & \text{if } k = 1 & \text{closed} \\ \chi & \text{if } k = 0 & \text{flat} \\ \sinh \chi & \text{if } k = -1 & \text{open} \end{cases} \quad (2)$$

Hereon, the speed of light, c will be set to one, so that a mass m and the corresponding energy mc^2 are both denoted by m .

The matter background will be assumed to be isotropic and homogeneous, described by a stress tensor T_ν^μ of a fluid in its rest frame. Thus in the $c = 1$ notation used here $T_\nu^\mu = \text{diag}(\rho, -p, -p, -p)$, where ρ is energy/matter density and p is pressure. The Einstein equations are (in terms of the Ricci tensor and scalar curvature which will not be needed subsequently)

$$(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}) = 8\pi GT_{\mu\nu} . \quad (3)$$

For a spacetime with metric and matter given above, and a cosmological constant Λ , these equations of motion yield

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3} . \quad (4)$$

Here, $\dot{R} = dR/dt$, etc. In addition the acceleration of $R(t)$ obeys

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad (5)$$

where p is pressure in the matter stress tensor above.

Write H_0 for the numerical value of H today, the Hubble constant, $H_0 = 100 h$ km/s/Mpc and R_0 for the value of the scale factor today. Then define

$$\begin{aligned} \Omega_k &= \frac{-k}{R_0^2 H_0^2} \\ \Omega_\rho &= \frac{8\pi G}{3H_0^2} \rho_0 \\ \Omega_\Lambda &= \frac{\Lambda}{3H_0^2} , \end{aligned} \quad (6)$$

constants corresponding to the values of these ratios today. The equation of motion, equation (4), implies

$$1 = \Omega_k + \Omega_\rho + \Omega_\Lambda . \quad (7)$$

The constants $\Omega_k, \Omega_\rho, \Omega_\Lambda$ can be used to rewrite equation (4) at a general time. Defining the scale factor $a(t) = R(t)/R_0$, $a = 1$ today, and the equation of motion (4) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2 R_0^2} + \frac{\Lambda}{3} \quad (8)$$

The a dependence of the terms proportional to the curvature k and Λ can be read off directly. The energy density ρ may have several components which scale differently as the universe expands. Nonrelativistic (“cold”) matter scales as the volume of the universe,

$$\Omega_{NR}(a) = \Omega_{NR}(R/R_0)^{-3} = \Omega_{NR}a^{-3} \quad (9)$$

while relativistic matter, such as radiation, redshifts,

$$\Omega_R(a) = \Omega_R a^{-4} . \quad (10)$$

In addition to these two familiar possibilities, other sorts of matter are possible which scale as other powers of a and are discussed in section six. Thus generally the energy density term Ω_ρ has several parts:

$$\Omega_\rho(a) = \Omega_{NR}(a) + \Omega_R(a) + \dots \quad (11)$$

As the universe expands, the energy density of the relativistic matter decreases faster than that of the non relativistic matter. In the standard cosmological scenario, earlier on the universe was radiation dominated, that is the relativistic energy density was the largest component of the matter energy density. Now non-relativistic matter makes up a larger fraction of the energy density (matter domination). For example, $\Omega_{R, \text{CMB}}$, the contribution from the cosmic microwave background (CMB) photons today, is about 10^{-5} , and is becoming increasingly negligible as a increases.

Including only relativistic and nonrelativistic matter for now, equation 8 above can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_{NR}}{a^3} + \frac{\Omega_R}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right), \quad (12)$$

with square root

$$\frac{1}{a} \frac{da}{dt} = H_0 \left(\frac{\Omega_{NR}}{a^3} + \frac{\Omega_R}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right)^{1/2} = H_0 E(z = a^{-1} - 1). \quad (13)$$

The generalization of \dot{a}/a and hence $E(z)$ to more general Ω_ρ is immediate once the dependence on the scale factor a is known.

The acceleration equation can be written in terms of Ω_{NR} , etc., as well, by including the different equations of state $p_a = w_a \rho_a$ for the different sorts of matter. If only relativistic and nonrelativistic matter is present, equation 5 becomes

$$\frac{\ddot{a}}{a} = -H_0^2 \left(\frac{\Omega_{NR}}{2} a^{-3} + \Omega_R a^{-4} - \Omega_\Lambda \right) \quad (14)$$

Equation 13 allows one to read off the effect of various possible combinations of relativistic and nonrelativistic matter, curvature and cosmological constant on the history of the universe. For example, one possibility (depending on magnitudes and signs) is that relativistic matter dominates for a small, then nonrelativistic matter, then curvature and finally the cosmological constant.

The behavior of the scale factor $a(t)$ for different $\Omega_\rho, \Omega_\Lambda$ are given in Felten & Isaacman (1986). The value of Ω_k determines whether the universe is spatially closed, open or flat, while models which expand forever or recollapse are determined by whether \dot{a} changes sign, passing through zero. One can see from equation (13) that if Ω_Λ is big enough, the universe will expand forever, eventually expanding exponentially fast.

II. EXPECTED Λ

Given that the cosmological constant may occur in the equations of motion, one can ask what value of Λ is expected. High energy particle theory, which in principle could include a theory of gravity, has no known way to determine the value of the cosmological constant from first principles. Many suggestions have been made, some of which are listed later in this section. However, cosmologically interesting scales for Λ are puzzlingly small in particle theory.

In particle theory, field quantization allows a zero point energy, the constant energy when all fields are in their ground state. This has the same effect as a cosmological constant. In the absence of gravity, only the difference between zero points of different systems can be measured, while the absolute value is unmeasurable. (The difference between the vacuum energies or zero points of different systems was demonstrated, for example, with the Casimir (1948) effect: Two parallel plate conductors provide boundary conditions for the vacuum between them. The vacuum energy depends on these boundary conditions. Changing the distance between the plates changes the boundary conditions, and thus the vacuum energy, resulting in a measurable force between the two plates.) However, Einstein's equations above react to the value of the vacuum energy itself, and the theoretical criteria for setting the absolute zero point are unclear.

The estimate of the vacuum energy in particle theory is reviewed in CPT. When fields are quantized in particle theory, they can be considered as a set of harmonic oscillators, and these each have an associated zero point energy. The total number of harmonic oscillators is determined by the high energy cutoff E_{max} of the theory. If there is no cutoff, that is, the naive field theory counting of degrees of freedom works to arbitrarily high energies and short scales, then E_{max} is infinite. Taking the sum of zero point energies and dividing by the volume of space gives the corresponding contribution to the constant energy density,

$$\rho_{vac} \sim \frac{E_{max}^4}{\hbar^3} . \quad (15)$$

An observed cosmological constant of order one corresponds to $E_{max} \sim 10^{-2} - 10^{-3} \text{eV}$. In contrast, it is not expected that naive field theory works at the scales where gravity is expected to become strong, so at the very least, a natural cutoff for particle theory is the scale of gravity, $E_{max,gravity} \sim 10^{28} \text{eV}$. There are also mechanisms within the context of field theory which can produce a cutoff, lowering E_{max} . For instance, supersymmetry relates particles of different spin. Supersymmetry can cause cancellations between zero point energies of different particles down to the scale where supersymmetry breaks, so that $E_{max,susy} \sim 10^{20} \text{eV}$. In most cases the expected particle physics scales for the cutoff give $E_{max} > 100 \text{GeV} = 10^{11} \text{eV}$, so that the vacuum energy density ρ_{vac} , proportional to Λ , obeys

$$\frac{\rho_{vac}^{obs}}{\rho_{vac}^{pred}} < \frac{(10^{-2})^4}{(10^{11})^4} = 10^{-52} \quad (16)$$

This huge disparity between the naively expected Λ and the value of Λ consistent with observations is, for particle physicists, ‘the cosmological constant problem.’ The relatively tiny value for a cosmologically relevant Λ seems very fine tuned from the particle physics point of view and so the prejudice in a large part of the particle physics community has been that it should be zero, by some mechanism not yet understood. An extremely small but nonzero Λ could perhaps be taken as some small effect perturbing around the preferred value of zero.

Before listing some particle theory mechanisms which have been suggested, it should be noted that many variants of inflation (Guth 1981, Linde 1982, Albrecht & Steinhardt 1982) correspond to an effective $\Lambda \neq 0$. (A notable exception is kinetic inflation, but this has many problems of its own, see Levin (1995) for discussion and references.) Inflation occurs when the scale factor accelerates, $\ddot{a} > 0$. A period of inflation in the early universe could have produced the observed homogeneity and isotropy seen today, as well as seeding density perturbations for structure formation. Slow roll inflation, for example, occurs when a field has approximately constant potential energy. This approximately constant potential energy behaves like a cosmological constant, eventually producing inflationary expansion. So in this inflationary model Λ was effectively nonzero in the past at some time. If there was an effective nonzero Λ in the past, by this or any other mechanism, any general prediction giving small or vanishing Λ now would have to allow for the earlier exception as well. Inflation generically drives $\Omega_k \rightarrow 0$, or $\Omega_\rho + \Omega_\Lambda \rightarrow 1$ (Peebles 1984, Turner, Steigman & Krauss 1984). It does not fix Ω_ρ or Ω_Λ separately.

There are several arguments, none as yet compelling, setting Λ to zero or a very small number. These include (based on a list by J. Lykken (1998)):

- Set the cosmological constant to its current value by fiat.
- Say that there are many possible universes with different values of Λ , but that any measurement we make has as the prior condition that we exist. This prior condition can be restated in terms of galaxies existing or other objects. Some references for these anthropic arguments are given in CPT. Recent developments include astrophysical versions producing Ω_Λ of similar size to Ω_ρ (for example Efstathiou 1995, Martel, Shapiro & Weinberg (1998) and references therein) and refinements of the probability measure for eternal inflation (for example Linde, Linde & Mezhlumian 1995, Vilenkin 1998 and references therein).
- Wormholes (nucleation of baby universes) might provide a dynamical potential which forces Λ to zero, although the arguments haven’t worked so far (Giddings and Strominger 1988, Coleman 1988, see CPT for more discussion).

- Some symmetry that isn't obvious might set Λ to zero, for example a string theoretic symmetry of the partition function (Moore 1987, Dienes 1990a,b) or a symmetry of some spacetime related to our spacetime via stringy symmetries (Witten 1995). Some modification, perhaps a small breaking of these symmetries, would be necessary to produce a small $\Lambda \neq 0$.
- A small value of Λ may appear from the ratio of two numbers of very different size (hierarchy) already present in particle physics. An example of a small ratio is the mass of a very light particle over the Planck mass, nonperturbative effects in particle theory also can provide small numbers of the form $e^{-const./\alpha}$ with α the coupling.
- The vacuum energy predicted above was found by counting the degrees of freedom in the context of field theory. In this case the number of modes is determined by the volume of space. This is in contrast to string theory, a candidate for combining gravity and quantum field theory, where the holographic hypothesis (Susskind 1994) states that a volume can be described by properties of its boundary. The number of degrees of freedom is then limited by the area of the boundary, not the volume of the space contained within. In the context of this hypothesis, the corresponding ρ_{vac} has been addressed by Banks (1995). An estimate of ρ_{vac} via a similar argument (using limits on black hole entropy to reduce the number of degrees of freedom) has also been given by Cohen, Kaplan, & Nelson (1998).

III. KINEMATICS

Kinematics with nonzero Ω_Λ is summarized in equation 8 above for the scale factor. This section will specialize to nonrelativistic matter, $\Omega_\rho = \Omega_{NR}$ for simplicity. The generalizations from including the other terms in Ω_ρ are more involved but immediate by substitution. With only nonrelativistic matter, equations (8, 13) become:

$$\frac{da}{dt} = H_0(\Omega_{NR}a^{-1} + \Omega_k + \Omega_\Lambda a^2)^{1/2}. \quad (17)$$

As a is increasing in an expanding universe, the factor multiplying Ω_Λ increases with time. If a keeps increasing, Ω_Λ eventually dominates. Conversely, at early times, $a \ll 1$, and so the contribution proportional to Ω_Λ is small. Even at the relatively recent redshift of $z = 10$ the contribution of Ω_Λ is suppressed by 10^{-3} relative to that of Ω_{NR} . Note that since the sum of $\Omega_\rho + \Omega_\Lambda + \Omega_k = 1$, changes in Ω_Λ can also be interpreted as changes in Ω_ρ and/or Ω_k .

Equation 17 can be used to calculate the age of the universe. The age is the integral of the time up to now in terms of the scale factor a :

$$H_0 \int dt = H_0 \int \frac{dt}{da} da = \int \frac{da}{(\Omega_{NR}a^{-1} + \Omega_k + \Omega_\Lambda a^2)^{1/2}} \quad (18)$$

Increasing Λ increases the age. This is one reason that cosmologists favoring $\Omega_k = 0$ (i.e. flatness, from inflation) wanted to introduce a cosmological constant when high H_0 observations were made. A high H_0 and lower limit on the age of the universe result in a lower limit on the left hand side of the above equation. Raising Ω_Λ raises the value of the right hand side to produce agreement. Other observations, including age measurements, have also been interpreted as requiring nonzero Λ even without demanding flatness (for example, Gunn & Tinsley 1975). In figure one below, one sees that H_0 times the age for a flat ($\Omega_k = 0$) universe increases as $\Omega_{NR} = 1 - \Omega_\Lambda$ decreases.

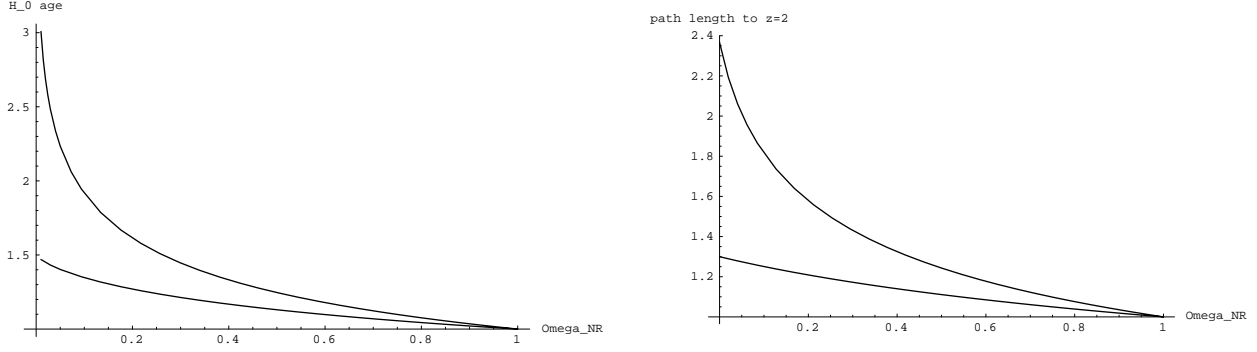


FIG. 1. Age and path length as a function of Ω_{NR} . On the left, the top curve shows H_0 times the age for a flat universe, $\Omega_\Lambda + \Omega_{NR} = 1$, plotted versus Ω_{NR} and normalized to one for $\Omega_{NR} = 1$. The lower curve shows the same quantities for an open universe, $\Omega_{NR} + \Omega_k = 1, \Omega_\Lambda = 0$. (The effects of a nonzero but small Ω_R are negligible in both.) Shown at right is the path length χ (equation (20)) of light back to redshift $z = 2$ relative to the path length for an $\Omega_{NR} = 1$ universe, again plotted as a function of Ω_{NR} . The upper curve is for a flat universe, $\Omega_{NR} + \Omega_\Lambda = 1$, and the lower curve is for an open universe, $\Omega_{NR} + \Omega_k = 1$.

The distance a light ray travels can be calculated as well. Light rays follow null geodesics where $ds^2 = 0$ so that $dt^2 = R_0^2 a^2 d\chi^2$. Then

$$\int R_0 d\chi = \int \frac{dt}{a(t)} = \int da \frac{1}{a} \frac{dt}{da} \quad (19)$$

and so the path back to scale factor a or redshift z is

$$\chi(a) = \int_a^1 \frac{da'}{R_0 a' \dot{a}'} = \int_0^{z=a^{-1}-1} \frac{dz'}{R_0 H_0 E(z')} . \quad (20)$$

This path length χ , for light going back to redshift $z = 2$, is shown above for a flat $\Omega_k = 0$ universe and for an open universe with $\Omega_\Lambda = 0$, as a function of varying Ω_{NR} . The path length is relevant for the measurements of lensing mentioned later. Another astrophysical quantity is the angular diameter distance d_A , the ratio of the proper size of an object to its apparent angular size. In this notation,

$$d_A = \frac{R_0 r(\chi)}{1+z} = \frac{1}{H_0 \sqrt{|\Omega_k|}} \frac{r(\chi)}{1+z} \quad (21)$$

and the luminosity distance, related to the known rest frame luminosity and the apparent flux, is $d_L = (1+z)^2 d_A$. (Recall that $c = 1$ in these units.)

One can also calculate the volume seen looking back to a given redshift. The comoving volume is

$$dV = R_0^3 r^2(\chi) d\chi d\Omega \quad (22)$$

and so using the equations above for $r(\chi)$ and $d\chi$ this can be calculated directly. The volume effects are relevant for measurements of number counts of objects such as gravitational lenses. More plots of the kinematic effects of changing Ω_Λ are found in CPT.

IV. STRUCTURE FORMATION

Structure formation proceeds by gravitational collapse of small amplitude density perturbations into larger and larger ones. A cosmological constant corresponds to an energy density which is smooth, that is, doesn't clump or

change on any scale. Thus, for a given Ω_k , increasing Ω_Λ decreases Ω_ρ , the matter available for gravitational collapse. Consequently structure grows more slowly in a universe with larger Ω_Λ for fixed Ω_k . As inflationary models can most easily provide $\Omega_k = 0$, comparisons often fix $\Omega_\Lambda + \Omega_\rho = 1$ to get intuition into the effects of Ω_Λ .

One can be slightly more precise by looking at the growth of density perturbations. The homogeneity of, for example, the CMB, tells us that at early times fluctuations had very small amplitude, of order parts in 10^{-5} . In this case linear perturbation theory is expected to work very well. At later times they can be followed into the nonlinear regime with semi-analytic methods or numerical simulations. Fluctuations on the largest scales have the smallest amplitude because they have only recently entered the horizon and thus only recently started to collapse. Hence on large scales, for example for galaxy redshift surveys, linear theory is useful for direct study of fluctuations.

For smaller scales, fluctuations have a larger amplitude and hence nonlinearities begin to be important. The scale for this crossover today is somewhere around $10 h^{-1}$ Mpc. In this case one can instead consider number densities. A good fit to the results of numerical simulations is to predict the number of regions with $\delta > \delta_c \sim 1.7$ and then model the collapse of these regions assuming spherical symmetry. This Press-Schechter (1974) method actually works well outside the expected range of validity; the excellent agreement of its results with simulations motivates its use. It can be considered as a very good analytic fit to simulation results. One application which will be discussed later is the rich cluster abundance.

The effects of a cosmological constant on linear perturbation theory is as follows. Over short distances, where the Newtonian limit is applicable, the equation of motion for density fluctuations of type A , $\delta\rho_A/\rho_A$ is (for example Padmanabhan 1993)

$$\ddot{\delta}_A + 2\frac{\dot{a}}{a}\dot{\delta}_A - \frac{v_A^2}{a^2}k^2\delta_A = 4\pi G \sum_B \rho_B \delta_B \quad (23)$$

where v_A is the sound speed. The sum on the right is over all components δ_B , but for ρ_Λ , $\delta_\Lambda = 0$ because the cosmological constant doesn't have any associated fluctuations. Nonzero Ω_k , curvature, also makes no contribution to the right hand side.

A simple case is when only one type of nonrelativistic energy density ρ_C dominates the fluctuations, $v_C \propto p_C/\rho_C = 0$. For redshifts where this Ω_{NR} dominates, equation (23) becomes

$$\ddot{\delta}_C + 2\frac{\dot{a}}{a}\dot{\delta}_C = 4\pi G\rho_C\delta_C \quad (\text{matter domination}) . \quad (24)$$

First ignore the expansion of the universe (the \dot{a}/a term) and the time dependence in ρ_C . Then the growing solution is $\delta_C \sim e^{\sqrt{4\pi G\rho_C}t}$, illustrating that increasing ρ_C increases the rate of fluctuation growth. This is to be expected since the right hand side is the change in energy density, the gradient of the gravitational potential. Including the expansion of the universe, the $\frac{\dot{a}}{a}\dot{\delta}$ term acts as a drag term to slow the growth of the perturbations, from exponential growth to power law, but the trend of faster growth with larger ρ_C remains. On the other hand, with only one sort of matter as assumed here, increasing Ω_Λ for fixed curvature Ω_k means decreasing ρ_C . Thus here increasing Ω_Λ means perturbations grow more slowly.

In the limit that the source term $\rho_C\delta_C$ is very small, the expansion of the universe prevents structure from growing. The limit $\rho_C = 0$ gives $\delta_C \sim \text{constant}$. The drag coefficient from the expansion of the universe is given by (equation (17) above)

$$\frac{\dot{a}}{a} = H_0\left(\frac{\Omega_{NR}}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\Lambda\right)^{1/2} , \quad (25)$$

while in the source term the nonrelativistic density ρ_C drops as a^{-3} . So if either Ω_k or Ω_Λ is nonzero, with increasing scale factor a , the drag term $\frac{\dot{a}}{a}\dot{\delta}_C$ will become larger than the source term $4\pi G\rho_C\delta_C$, and structure will eventually stop growing. The extreme case of this is when the cosmological constant dominates at late times, making the universe inflate. Then, regions which have broken away via gravitational collapse will continue to evolve separately (hence the term breaking away), but these regions will be separated from each other more and more as time goes on.

Equation (24) has a formal solution. In this equation, make the substitution

$$\frac{d}{dt} = \dot{a} \frac{d}{da} = aH \frac{d}{da}, \quad (26)$$

define $\delta' = d\delta/da$, etc., and define $\delta(a)$ is implicitly by $\delta(t)$ and vice versa via $a(t)$. Dropping the subscript C , the fluctuation equation (24) then becomes

$$a^2 H^2 \delta'' + a(3H^2 + aHH')\delta' = 4\pi G \frac{\rho_{NR}}{a^3} \delta, \quad (27)$$

with solution (Heath 1977)

$$\delta(a) = \frac{5}{2} H_0^2 \Omega_{NR} \frac{\dot{a}}{a} \int_0^a \frac{d\tilde{a}}{\tilde{a}^3}. \quad (28)$$

This has been normalized to give $\delta(a) \sim a$ in the case $\Omega_{NR} = 1, \Omega_\Lambda = 0$. The behavior of the solution $\delta(a)$ is shown in Figure 2 below for two cases, $\Omega_{NR} = 1, \Omega_\Lambda = 0$ and $\Omega_{NR} = .3, \Omega_\Lambda = .7$. The relative size of the fluctuations in different cosmologies today is given by $\delta(a = 1)$.

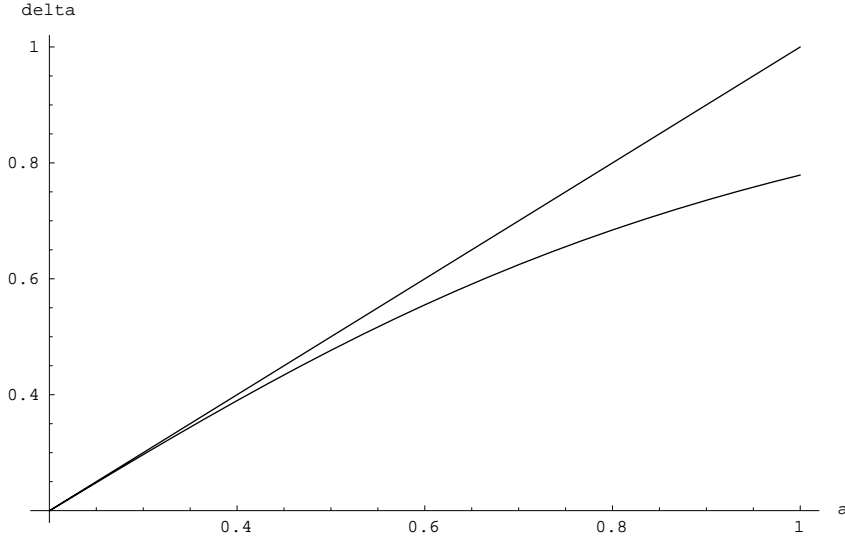


FIG. 2. Growth of perturbations as a function of a for an $\Omega_{NR} = 1, \Omega_\Lambda = 0$ model (where $\delta(a) \sim a$) and for a $\Omega_{NR} = .3, \Omega_\Lambda = .7$ model (lower curve).

Thus, in summary, both curvature and the cosmological constant have no fluctuations, corresponding to smooth components, $\delta_{curv} = \delta_\Lambda = 0$. Consequently fluctuations grow more slowly when there is a positive cosmological constant, or if there is curvature $\Omega_k > 0$.

Peculiar velocities also respond to gravitational potentials and hence the growth of perturbations, but the rate of growth and peculiar velocities today ($z \sim 0$) are only very weakly dependent upon the cosmological constant, as are many other dynamical measurements (Lahav et al 1991, CPT). For example, the growth of perturbations at the present epoch goes as $\sim \Omega_{NR}^{0.6} + \frac{1}{70} \Omega_\Lambda (1 + \Omega_{NR})$.

V. MEASUREMENTS

Recently there have been significant observational advances relevant to measuring the cosmological constant. These include the following.

Gravitational Lenses:

Gravitational lensing of quasars by intervening galaxies measures the volume of space back to a given redshift, assuming constant comoving density of lensing objects. The volume back to a given redshift (derived from equation (22)) has a strong dependence on the cosmological constant in a flat universe (Fukugita et al 1990, Turner 1990, much of the analysis was implicit in Gott et al 1989). The lens density relative to the fiducial case $\Omega_{NR} = 1, \Omega_\Lambda = 0$ is (Fukugita et al 1992, CPT):

$$P_{lens} = \frac{15}{4} \left(1 - \frac{1}{\sqrt{1+z_s}}\right)^{-3} \int_0^{z_s} dz \frac{(1+z)^2}{E(z)} H_0^2 \left[\frac{d_A(0,z)d_A(z,z_s)}{d_A(0,z_s)} \right]^2 \quad (29)$$

for a source at redshift z_s . Recall that $c = 1$ so that $H_0 R_0$ is dimensionless. The angular distance $d_A(z_1, z_2)$ between redshifts z_1, z_2 is the generalization of equation (21):

$$d(z_1, z_2) = \frac{R_0}{(1+z_2)} \text{sinn} \left\{ \int_{z_1}^{z_2} dz \frac{1}{R_0 H_0 E(z)} \right\} \quad (30)$$

In Figure 3, P_{lens} is plotted for a flat ($\Omega_k = 0$) universe for sources at $z_s = 2$. The effect of a large Λ in a flat universe is very strong for $\Omega_{NR} < .3$, i.e. $\Omega_\Lambda > .7$. The results of integrating the above for a range of different Ω_k are found in CPT.

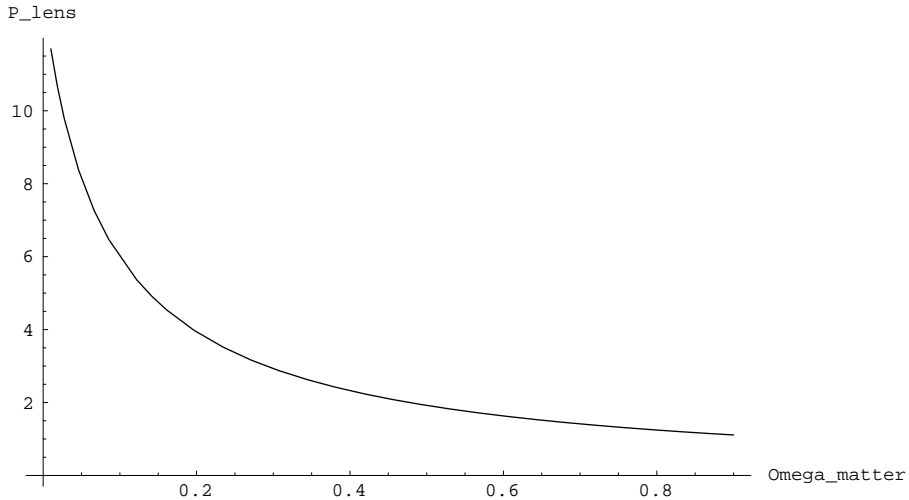


FIG. 3. Number of lenses expected for sources at $z=2$ in a flat universe with varying $\Omega_{matter} = \Omega_{NR}$, relative to the number expected for $\Omega_{matter} = 1$.

The current situation with the data is as follows. For $\Omega_k = 0$, analysis (Kochanek 1993, 1996; Maoz & Rix 1993) of surveys for multiply imaged quasars give a 2σ upper limit of $\Omega_\Lambda < .66$. There are statistical errors because of the small number of lensed quasars in the survey and uncertainties in the local number counts of galaxies by type. Possible sources of systematic errors (for a full list of references see Falco, Kochanek & Munoz 1998) include extinction, galaxy evolution, the quasar discovery process and the model for the lens galaxies. Using radio selected lenses to reduce possible systematic errors associated with extinction and the quasar discovery process, Falco et al find $\Omega_\Lambda < .73$ at 2σ . Combining the radio and optical data they find $\Omega_\Lambda < .62$ at 2σ for their most conservative model. The errors should improve significantly with the upcoming observations, including for example the CASTLe survey (the

CfA-Arizona Space Telescope Lens Survey)¹, which will have an HST survey to measure redshifts of lens galaxies, and the completion of the CLASS (Cosmic Lens All-Sky Survey)² radio lens survey.

Cluster abundance:

In the previous section it was argued that fluctuations grow more quickly for larger Ω_ρ . Thus, fixing Ω_k and increasing Ω_Λ , perturbations will grow more and more slowly. Galaxy clusters are rare objects on large scales which have only recently gone nonlinear, and thus many of the complications due to nonlinearities may be expected to be less important. Consequently, normalizing to the rich cluster abundance seen today at $8 h^{-1}$ Mpc, σ_8 , and fixing Ω_k , a higher Ω_Λ universe has earlier cluster formation (i.e. at higher redshift).

An example of the different number of clusters expected as a function of cosmology is shown below.

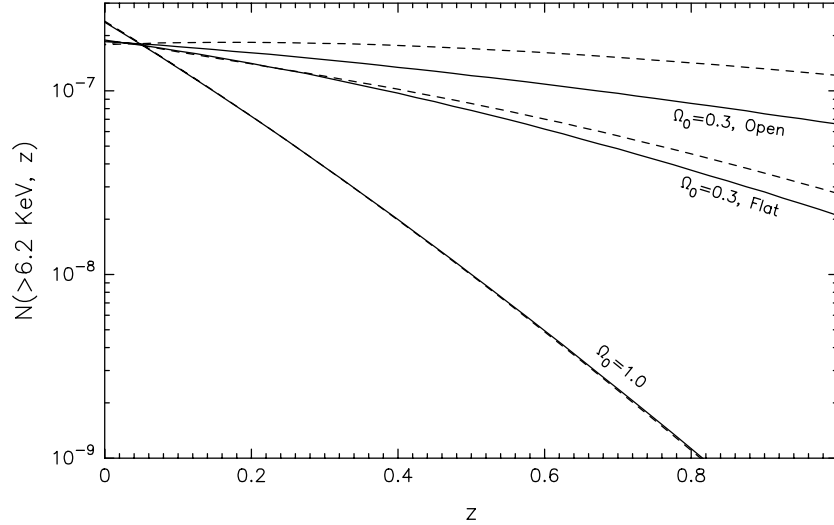


FIG. 4. Expected redshift evolution of $N(> 6.2\text{keV}, z)$, the comoving number density of galaxy clusters with an X-ray temperature $k_B T > 6.2\text{keV}$ at redshift z , normalized to produced the observed $N(> 6.2\text{keV}, 0.05)$. The cluster virialization redshift z_c was estimated using a method of Lacey & Cole (1993, 1994) for the solid lines, and z_c was taken to coincide with the cluster redshift for the dashed lines. The $\Omega_0 = 0.3$ flat model has $\Omega_\Lambda = 0.7$. Courtesy of Viana and Liddle.

To compare observations with theory, Press-Schechter or N-body calculations are used to calculate the expected mass distribution, and then the mass is related to the X-ray temperature. The relation between the mass and the X-ray temperature is one of the major uncertainties. The numerically calculated cluster number evolution can then be compared to current data. A wide range of results are found in Eke et al (1998), Viana & Liddle (1998), Reichart et al (1998), Blanchard & Bartlett (1997), Henry (1997), Fan, Bahcall & Cen (1997), Gross et al (1997), Henry (1997) and references therein.

The deepest complete X-ray sample available is from the Einstein Medium Sensitivity Survey (EMSS)³. The statistics and uncertainties (both in modeling and in the data) are not yet well enough under control to determine Ω_ρ conclusively (Colafrancesco, Mazzotta & Vittorio 1997, Viana & Liddle 1998), although the mere presence of very

¹<http://cfa-www.harvard.edu/castles>

² <http://dept.physics.upenn.edu/~myers/class.html>

³ http://www.tac.dk~lars_c/gammabox/doc/emss.html

high ($z > 1$) redshift clusters has been argued sufficient to rule out $\Omega_\rho = 1$ (for example, Gioia 1997, Bahcall & Fan 1998b and references therein).

Several new surveys are underway in the X-ray and optical, and upcoming satellites such as AXAF (the Advanced X-ray Astrophysics Facility)⁴ and XMM (X-ray Multi-mirror Mission)⁵ should allow better determinations of cluster properties. Thus observational data should markedly improve in the very near future.

Arcs:

The number of galaxy clusters at a given time also affects the number of strong gravitational lenses (i.e. arcs). Recently Bartelmann et al (1997,1998, see references therein for earlier work) analyzed the number of lenses expected in different cosmologies. For background sources at $z_s \sim 1$, clusters at redshifts $0.2 \leq z_c \leq 0.4$ are the most efficient lenses, with only weak dependence on cosmology. More lenses are expected for lower Ω_ρ (and hence higher Ω_k and Ω_Λ) because in this case the clusters form earlier and thus are more likely to be present to act as lenses. In addition, they argue that clusters forming at an earlier time are expected to be composed of more compact subclusters and thus be more efficient at lensing.

Using two cluster simulation methods and varying the cosmological parameters (nine simulations in all), Bartelmann et al (1997, 1998) found an order of magnitude more arcs are produced in a flat $\Omega_\Lambda = 0.7$ than in a flat $\Omega_\Lambda = 0$ universe. (For an open $\Omega_\rho = 0.3$, $\Omega_\Lambda = 0$ universe they expect an additional factor of ten more.) The number of arcs estimated from extrapolating the EMSS (Einstein X-ray Extended Medium Sensitivity Survey) arc survey to the whole sky is about 1500-2300, while the number of arcs expected from their simulations is

$$N_{arcs} \sim \begin{cases} 2400 & \Omega_\rho = .3 & \Omega_\Lambda = 0 \\ 280 & \Omega_\rho = .3 & \Omega_\Lambda = .7 \\ 36 & \Omega_\rho = 1 & \Omega_\Lambda = 0 \end{cases} \quad (31)$$

As with most observations described here, the data is expected to improve significantly.

Radial/Transverse distance ratios:

A method which shows much promise for investigating Ω_Λ in the future is related to the distortion of quantities which are intrinsically isotropic (Alcock & Paczynski 1979). Transverse and radial distances have a different dependence on Ω_Λ , if they are equal this is a condition on cosmological parameters. In more detail, from the metric, equation (1), the comoving distance between two objects is

$$d\chi^2 + r^2(\chi)[d\theta^2 + \sin^2 \theta d\phi^2] = \left(\frac{d\chi}{dz}\right)^2 dz^2 + r^2(\chi)[d\theta^2 + \sin^2 \theta d\phi^2], \quad (32)$$

where $d\chi$ has been rewritten as $\frac{d\chi}{dz}dz$. With an isotropic spacetime, something which is intrinsically isotropic should be measured to have the same size in all directions. For example, the measured distance in the redshift (radial) direction should be the same as the measured distance in the angular (θ) direction:

$$\frac{d\chi}{dz}dz = r(\chi)d\theta. \quad (33)$$

As $d\chi/dz = 1/(R_0 H_0 E(z))$ from equation (20), and $r(\chi)$ is given by equations (2, 20), one expects for an intrinsically isotropic object that

$$r(\chi)R_0 H_0 E(z) = \frac{dz}{d\theta}. \quad (34)$$

⁴ <http://asc.harvard.edu/>

⁵ <http://astro.estec.esa.nl/XMM/xmm.html>

The left hand side is calculable for a given model while the right hand side is measured. The left hand side can also be written in terms of the angular diameter distance d_A given earlier. The notation f/g is used in Phillipps (1994) and subsequent papers.

Although galaxy clusters, the original candidate for isotropic objects, are not intrinsically isotropic enough for this test, the correlation function of, for instance, quasar pairs (Phillips 1994), is expected to be intrinsically isotropic when averaged over pairs. The feasibility of using this particular test with the Sloan Digital Sky Survey⁶ and the Two Degree Field Survey⁷ has been analyzed by Popowski et al (1998), other suggested uses of this test are also referenced and discussed therein.

Supernovae:

Type Ia supernovae are considered to be calibrated candles (once a width-brightness relation is applied, their absolute magnitude M is believed to be known). Their apparent magnitude m is given by

$$m - M = 5 \log_{10} \frac{d_L}{\text{Mpc}} + 25 \quad (35)$$

where d_L , the luminosity distance, is given by

$$d_L = R_0(1+z) \int_0^z \frac{dz'}{R_0 H_0 E(z')} \quad (36)$$

Thus the curve for $d_L(z)$ versus z depends upon cosmological parameters.

Using a method pioneered by Perlmutter et al (1995), two groups (Perlmutter et al 1998, Riess et al 1998 and references therein) have been steadily finding and analyzing supernovae for the past couple of years. The data is accumulating rapidly. As of this writing, $d_L(z)$ versus z results for several dozen high redshift supernovae have been announced, and light curves of many others are being followed, several of which will have HST observations. The combined results have so far indicated high confidence for a positive nonzero cosmological constant. For the most recent information their web pages can be consulted⁸.

Possible systematic errors which have been studied include extinction and evolution, selection effects, weak lensing, the width-brightness relation used for calibration and sample contamination. In particular, there is very little reddening seen in either the far or near supernovae. There are models which predict this, for example the Monte Carlo analysis by Hatano et al (1997) on observability of supernovae predicts "the probability distribution of extinction for type Ia supernovae to be strongly peaked near zero." As so much relies upon SN Ia being standard candles, the question of evolution is crucial as well. Spectra of the high and low redshift supernovae have been compared with good agreement. The distribution of faint and bright supernovae appears similar at both high and low redshift. Many of the uncertainties now come from the low redshift samples. It is expected that there will be improvements in understanding the calibrations as the number of supernovae Ia found in galaxies where there are Cepheids increases (Kirshner 1998).

CMB anisotropies:

The study of CMB anisotropies provides a useful complement to the supernovae results. The supernovae results at redshift $z \sim .5$ measure $\Omega_\rho - \Omega_\Lambda$. In contrast, the CMB anisotropy measurements yield a power spectrum as a function of angle. Thus, positions of features in the spectrum, in particular the first doppler peak (White and Scott 1996), measure the angular diameter distance (equation (21), which is a function of the weighted sum of $\Omega_\rho + \Omega_\Lambda$. Combining the results from both types of studies is a promising tool (White 1998, Tegmark, Eisenstein & Hu 1998, Tegmark et al 1998b, Garnavich et al 1998). The current best fit to the existing data is (Lineweaver 1998)

⁶<http://www.astro.princeton.edu/BOOK>

⁷<http://meteor.anu.edu.au/colless/2dF>

⁸ <http://www-supernova.lbl.gov/> and <http://cfa-www.harvard.edu/cfa/oir/Research/supernova/HighZ.html>

$$\Omega_\Lambda = .62 \pm .16, \quad \Omega_\rho = .24 \pm .10 \quad (37)$$

where the errors denote 68.3% confidence level. A likelihood plot of supernovae and CMB data is shown in figure five below. The current situation is on the left, while the expected precision for a particular high Ω_Λ model after the flight of the Planck surveyor⁹ is shown at right.

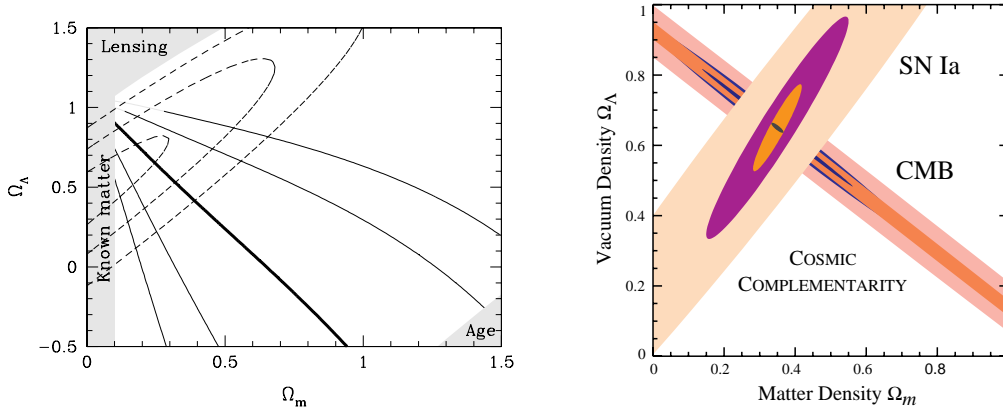


FIG. 5. Cosmic complementarity: likelihood in the Ω_m - Ω_Λ plane for fitting both SN-Ia and CMB data. On the left is compilation of the current data, courtesy M. White. The dashed lines are 1σ , 2σ and 3σ contours for fitting the SN-Ia results, the solid lines are the CMB results. The thick solid contour denotes the peak of the likelihood found by Hancock et al (1998), and the two contours to either side represent conservative $\pm 1\sigma$ and $\pm 2\sigma$ values. The shaded areas are ruled out by other constraints as indicated. On the right is shown, courtesy Eisenstein & Hu, the precision expected with CMB data from upcoming Planck satellite mission and ‘optimistic’ SN-Ia data, for a model with $\Omega_{NR} = 0.35$, $\Omega_\Lambda = 0.65$. One σ confidence regions are shown and the combined data has an error region the size of the overlap. The central bar gives the errors expected with polarization included. For more discussion see Tegmark et al (1998b).

As the CMB anisotropy data will be improving in upcoming years, so will these tests. The CMB can also be used in conjunction with measures of Ω_ρ , such as mass/light ratios (see this recent review by Bahcall & Fan (1998a)) or peculiar velocities of galaxy clusters. For example, low Ω_ρ combined with CMB evidence for a flat universe would support $\Omega_\Lambda \neq 0$.

Other measures:

There are several other tests, including number counts, QSO line statistics, galaxy mergers, weak lensing surveys (for example Van Waerbeke, Bernardeau & Mellier 1998 and references therein), etc., CPT discuss several of these. Some tests, such as number counts, are now understood to be less reliable than earlier believed due to evolutionary effects.

VI. MAYBE NOT Λ

Although a cosmological constant has been suggested as providing the missing energy density in the universe, this is not the only possible explanation of the observations. As mentioned earlier, the extremely small but nonzero cosmological constant Λ consistent with current data is theoretically unnatural given the current understanding of particle physics. This gives rise to the question of whether the observations can be explained by some other form of

⁹<http://astro.estec.esa.nl/Planck/>. Planck will also be a source of relevant data for other tests, for example lenses (Blain 1998 and references therein).

energy density as well. In some scenarios this other energy density could replace nonzero Λ entirely, so that $\Omega_\Lambda = 0$. Other forms of energy density have been studied in the past, in particular by those wishing to combine the simplest inflationary prediction $\Omega_k \rightarrow 0$ with early observations which favored small Ω_{NR} and Ω_Λ . Other motivations include the observations that the galaxy power spectrum amplitude is too high on all scales for Λ models (for example White & Scott 1996, Coble et al 1996).

Thus one can postulate some additional energy density scaling as

$$\Omega_X \sim a^{-3(1+w)} \tag{38}$$

with $w = p_X/\rho_X$. For simplicity (again generalization is straightforward), take only one type of Ω_X in addition to the energy densities already considered. The limit $w \rightarrow -1$ gives the cosmological constant. This additional energy density is called by several names: quintessence, generalized dark matter, or, if smooth, X-matter. It is often specified by a field theoretic model, and includes models with a ‘time dependent cosmological constant’. Many candidates for quintessence have been proposed over the years; several of the papers previous to 1992 are described and summarized in CPT, some more recent descriptions include Coble et al (1996), Ferreira & Joyce (1997a, 1997b), Anderson & Carroll (1997), Carroll (1998), Zlatev, Wang & Steinhardt (1998). See these papers for additional references and discussion. The matter in these field theories sometimes has special properties. For example, when a scalar field ϕ in a shallow potential is used to provide the extra matter, the field excitations in the simplest case are nearly massless, $m_\phi \sim 10^{-33}\text{eV}$. (This should be compared to limits on the photon’s rest mass, which are in the range $10^{-18} - 10^{-27}$ eV (Nieto 1993).) Of course, just as for the cosmological constant, this small number could be attributed to the ratio of two numbers with very different sizes already present in particle physics, for example a scalar depending on the tiny ratio of neutrino mass/Planck mass (Hill & Ross 1988). And more generally, there are models going beyond the simplest case.

The equation of state specified by w can be constant or time varying and is related to the potential in the case of models based on a scalar field. For tangled strings (Vilenkin 1984, Spergel and Pen 1997) and textures (Davis 1987, Kamionkowski & Toumbas 1996) $w = -1/3$.

Including this Ω_X in Einstein’s equations, equation 8 above becomes

$$\frac{da}{dt} = H_0 \left(\frac{\Omega_{NR}}{a} + \frac{\Omega_R}{a^2} + \Omega_k + \Omega_\Lambda a^2 + \frac{\Omega_X}{a^{1+3w}} \right)^{1/2} \tag{39}$$

The kinematic analysis from section three can be repeated immediately. In order to meet current age and Hubble constant measurements (Turner & White 1997, Chiba et al 1998a, Caldwell et al 1998a) $w < 0$ is preferred. The general evolution of $a(t)$ for different equations of state is given in a recent paper by Overduin and Cooperstock (1998).

To consider effects on structure formation, the behavior of fluctuations must be specified in addition to the background equation of state. For example, a tangled network of light strings will be very rigid, and hence approximatedly smooth on subhorizon scales. If Ω_X is smooth on subhorizon scales, there is less matter available to fall into density perturbations, and so in this case structure grows more slowly. More generally, a three parameter stress energy tensor for this matter has been given in Hu (1998), for the case when the stress fluctuations are derived from density and velocity perturbations.

Cosmological tests such as CMB predictions, cluster abundance, supernovae results etc. have been analyzed for a variety of forms of X-matter. Some recent papers include Turner and White (1997), Caldwell et al (1998a, 1998b), White (1998), Wang & Steinhardt (1998), Huey et al(1998), Chiba et al (1998a, 1998b), Hu et al (1998), Garnavich et al (1998) and references therein. As there are many parameters which can be tuned in these generalized dark matter theories, it is possible to find variants which currently fit observations just as well as the case where cosmological constant is actually constant. Just as a search a theoretical explanation of the size of the cosmological constant is underway, compelling theoretical explanations for the presence of generalized dark matter candidates are also lacking. Currently it is a question of aesthetics as to which type of matter, the cosmological constant or generalized dark matter, is a more reasonable way to provide the apparently missing energy density.

VII. SUMMARY

Before summarizing it is worthwhile to remind people of the future of a universe with large positive cosmological constant. In this case, the universe will expand faster and faster, and eventually enter a stage of inflation, approaching exponential expansion $a \sim e^{\sqrt{\Lambda/3}t}$. Number densities will drop and the universe will become more and more homogeneous as time goes on. If there is no mechanism to change Λ , that is, if the cosmological constant is truly constant, then this expansion will continue indefinitely. If, instead, the cosmological ‘constant’ decreases at some point, then reheating, the filling of the universe with particles (field fluctuations) may take place, just as happened in our past if inflation occurred.

To summarize, there is now strong evidence for accelerated expansion of the universe from supernovae results. The supernovae and other measurements are improving rapidly. Some evidence which in the past was used to support $\Omega < 1$ is $\Omega_{NR} < 1$ evidence, and hence consistent with this picture. Besides or in place of a cosmological constant, this observational evidence might also be consistent with matter having some unusual equation of state. As such an equation of state has several parameters, many of the tests (some described above) narrow the parameter space rather than ruling out this extra form of matter. Rapid improvement in the quality of the data is expected over the next few years.

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