Converting heliocentric circular orbits to geocentric orbits

Parameters for an outer planet

The heliocentric parameters

We will consider the conversion of simple circular heliocentric orbits to geocentric orbits with a goal to find the deferent radius and epicycle radius and the corresponding periods in the geocentric reference frame. From the geometry in Fig. 1 we see the displacement from the earth to the planet is given by

$$\vec{r} = \vec{r}_p - \vec{r}_e \tag{1}$$

and the magnitude of the displacement is obtained from

$$r^2 = r_p^2 + r_e^2 - 2\vec{r}_e \cdot \vec{r}_p = r_p^2 + r_e^2 - 2r_e r_p cos(\theta_e - \theta_p) \tag{2}$$

The earth and planet move at constant angular velocities ω_e, ω_p

The geocentric parameters

In the geocentric system it is assumed that the planet moves around the epicycle at constant angular velocity ω_C. The angle of anomaly, $\alpha = \alpha_0 + \omega_C t$, is measured between the epicycle radius R_C and the deferent radius R_D, See Fig. 2. Note that zero angular velocity in the epicycle corresponds to R_C always remaining parallel to R_D. The center of the epicycle, C, lies on the deferent circle and the radius of the deferent makes an angle $\beta = \beta_0 + \omega_D t$ with repect to the x axis. The deferent radius rotates at constant angular velocity ω_D. From Fig. 2 we see that the displacement between the earth and planet is

$$\vec{r} = \vec{R}_D + \vec{R}_C \tag{3}$$

Consequently the displacement between the earth and the planet is obtained from

$$r^2 = (\vec{R}_D + \vec{R}_C) \cdot (\vec{R}_D + \vec{R}_C) = R_D^2 + R_C^2 + 2R_D R_C cos(\alpha) \tag{4}$$

We note that the two equations, 2 and 4, refer to the same displacement between the earth and the planet. Consquently

$$R_D^2 + R_C^2 + 2R_D R_C cos(\alpha) = r_p^2 + r_e^2 - 2r_e r_p cos(\theta_e - \theta_p). \tag{5}$$

In equation 5 we have a time independent part and a time dependent part. In order for equation 5 to be true at all times the time independent parts must be equal and the time dependent parts must be equal.

$$R_D^2 + R_C^2 = r_p^2 + r_e^2 \tag{6}$$

and

$$2R_D R_C cos(\alpha) = -2r_e r_p cos(\theta_e - \theta_p) \tag{7}$$

The radii are always positive so in equation 7 the sign difference must be related to the angles. Also $\theta_e - \theta_p = (\omega_e - \omega_p)t$ so

$$\alpha = \alpha_0 + \omega_C t = (\omega_e - \omega_p)t \tag{8}$$

From equation 8 we conclude that $\alpha_0 = \pi$ and $\omega_C t = (\omega_e - \omega_p)t$. From equation 7 we can show that there are two possible solutions.

$$R_D = r_p, R_C = r_e, (solution1) \tag{9}$$

or

$$R_D = r_e, R_C = r_p, (solution2) \tag{10}$$

Now we decide which solution from eqns. 9, 10, corresponds to the physically accessible solution. For this we investigate Fig. 3. Consider diagram (a) which shows the heliocentric arrangement of the sun-earth-planet (sep) for the distance of closest approach between the earth and an outer planet. There must be such a configuration and we define the x axis as the line connecting these three colinear points. The earth must lie between the sun and the planet. In the case of diagram (b), the geocentric solution 1, $R_D = r_p$ and $R_C = r_e$. Here the arrangement of sep is the same as in the heliocentric reference frame. We also note that $\alpha = \pi$. In the case of diagram (c), the geocentric solution 2, $R_D = r_e$ and $R_C = r_p$. Here the sun lies between the earth and the planet. Changing the reference frames from heliocentric to geocentric can not change the physical arrangement of the sun-earth-planet configuration. Hence, for an outer planet viewed from earth we require solution 1.

Next we need to determine the angular velocity of the deferent (point C in Fig. 4). We consider the time dependence of the x component of r, r_x. Write the x component in both heliocentric and geocentric terms.

$$r_x = r_p cos(\theta_p) - r_e cos(\theta_e) = r_p cos(\omega_p t) - r_e cos(\omega_e t). \tag{11}$$

$$r_x = R_D cos(\beta) + R_C cos(\alpha + \beta) \tag{12}$$

In eqn 12 we have $\beta = \beta_0 + \omega_D t$ and $\alpha = \alpha_0 + \omega_C t = \pi + (\omega_e - \omega_p)t$. We use these facts in eqns 11 and 12.

$$r_p cos(\omega_p t) - r_e cos(\omega_e t) = R_D cos(\beta) + R_C cos(\alpha + \beta) = r_p cos(\beta) + r_e cos(\alpha + \beta) \tag{13}$$

$$r_p cos(\omega_p t) - r_e cos(\omega_e t) = r_p cos(\beta_0 + \omega_D t) + r_e cos(\pi + (\omega_e - \omega_p)t + \beta_0 + \omega_D t) \tag{14}$$

In eqn 14 we compare the terms multiplying r_p and see that consistent results can be obtained with the following set of choices: $\beta_0 = 0, \omega_D = \omega_p$. We summarize the parameter choices for an outer planet.

$\boxed{R_D = r_p, R_C = r_e, \omega_D = \omega_p, \omega_C = \omega_e - \omega_p}$

Parameters for an inner planet

If we consider the motion of the earth from the perspective of the outer planet this must give the same value of r and R_D, R_C. The argument assigning the deferent radius R_D essentially assigns R_D to the larger of the two orbits. This argument depended on conserving the order sep for the colinear configuration

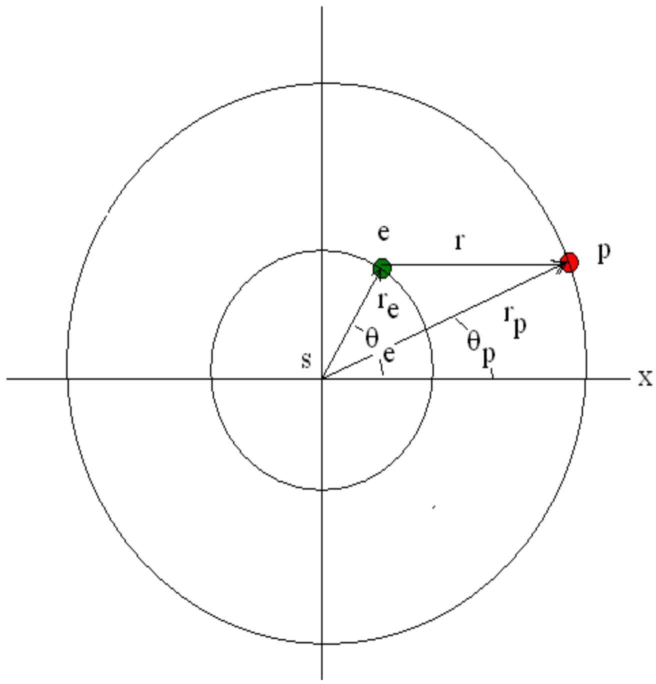

Figure 1: Circular orbits in the heliocentric reference frames. The earth and planet are at radii r_e, r_p. The radius of the earth is at an angle θ_e and the planet's radius is at angle θ_p. The sun, s, is the origin of coordinates.

leading to smallest distance of approach. This argument also applies to the inner planet as viewed from earth. All we are doing is interchanging the labels p for e for an inner planet. So in either case when we convert to a geocentric reference frame we can summarize the results.

$\boxed{R_D = \text{larger orbit, } R_C = \text{smaller orbit, } \omega_D = \text{ angular velocity of the larger orbit}}$

$\boxed{\omega_C = \text{ absolute value of the difference } \omega_e - \omega_p}$

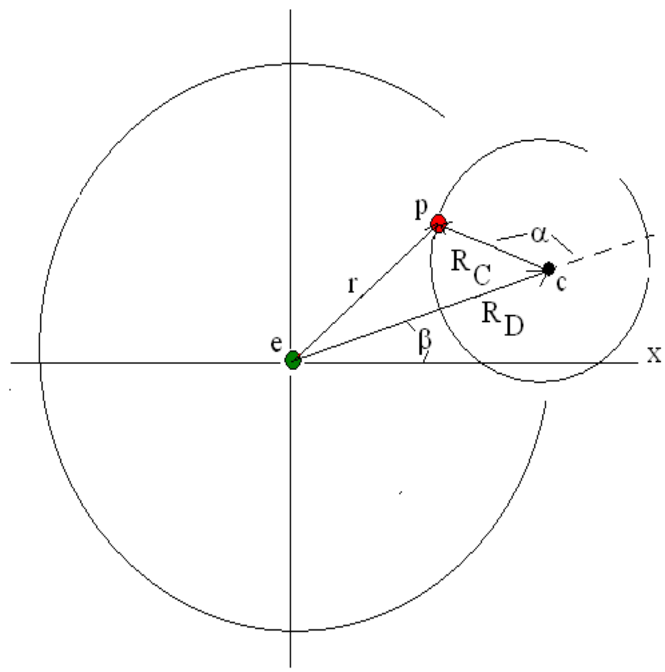

Figure 2: Circular orbits in the geocentric system with a deferent orbit, R_D, and an epicycle radius, R_C. The angle of anomaly, as per the definition from antiquity, is α. The center of the epicycle, c is on the deferent. The deferent radius R_D makes an angle β with respect to the x axis. The earth, e, is at the center of coordinates.

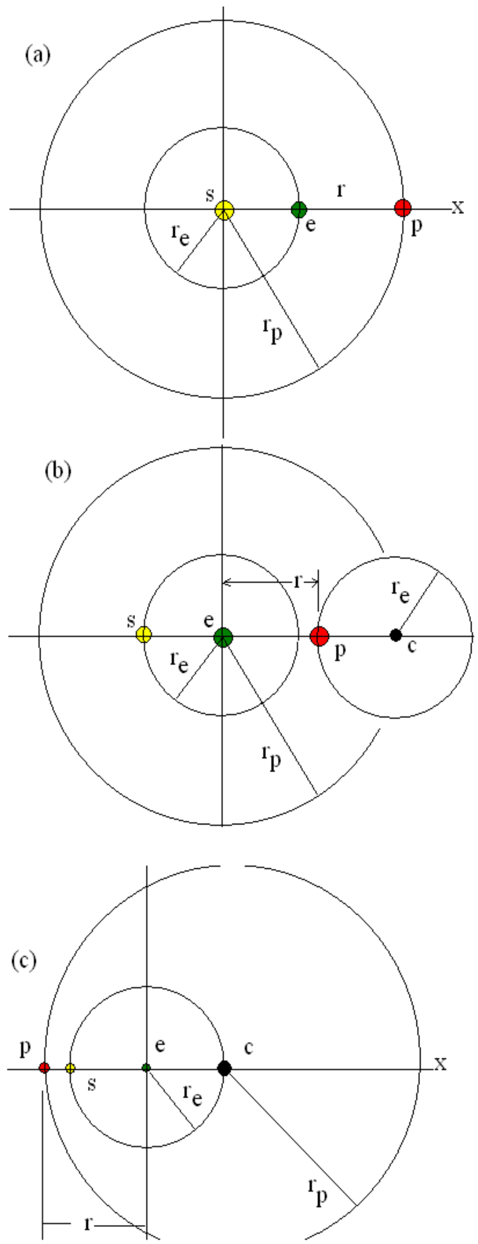

Figure 3: (a): The heliocentric reference frame for distance of closest approach bewtween earth and the planet. (b):Geocentric solution 1 where R_D=r_p and $R_C = r_e$. (c): Geocentric solution 2 where $R_D = r_e$ and $R_C = r_p$.

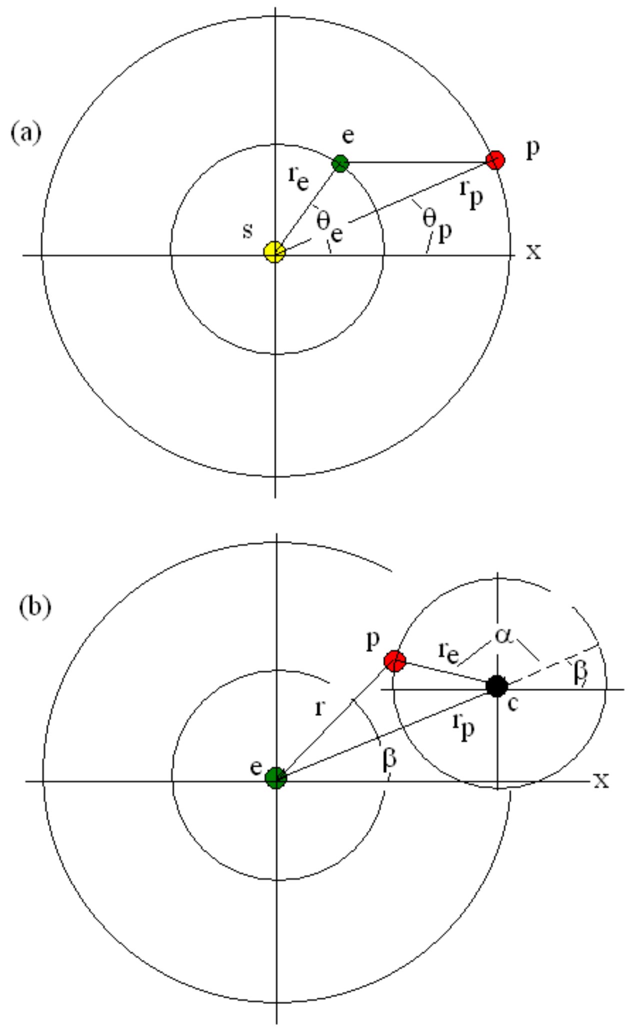

Figure 4: (a): The heliocentric reference frame used to calculate the x axis component of r, r_x, $\vec{r} = \vec{r}_p - \vec{r}_e$. (b): The geocentric reference frame used to calculate the x axis component of r, r_x, $\vec{r} = \vec{R}_D + \vec{R}_C$. Solution 1 is used, $R_D = r_p, R_C = r_e$