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PURPOSE:

These notes are intended to assist students in instructional science laboratories, especially in physics, and to eliminate the need to repeat this material in each laboratory manual of a course sequence.

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ESTIMATING ERRORS

INTRODUCTION

When the result of a scientific measurement is to be communicated, in a publication or even in a physics lab report, three pieces of information are essential:

the VALUE for example 54.3

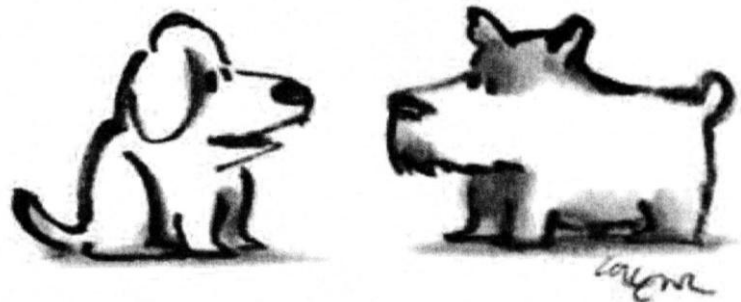
its UNITS " " cm.

its ESTIMATED UNCERTAINTY
examples ± 0.2 cm, $\pm 0.4\%$

and a fourth piece is common:

COMPARISON to values obtained by other methods or other people.

This guide is focused on the third item above, the procedures for assigning an appropriate estimated uncertainty to a result. The words "error" and "uncertainty" are used interchangeably except where specially noted. The phrase "human error", as a euphemism for mistake, is, however, a forbidden term in lab reports.



"I attribute it to human error. But then I attribute everything to human error."

Understanding the potential sources of errors in an experiment often dictates the design of the apparatus and the method or procedure. Furthermore, in many cases simply identifying and understanding the source of an error will lead to ways of eliminating it.

STATEMENT OF UNCERTAINTY ESTIMATES

There are several accepted, and to some extent equivalent ways of expressing the uncertainty in a result. The three most suitable to physics courses are:

Significant Figures. The way in which the value is stated should in itself be a crude indication of its precision. Only the last stated figure should be in doubt, and the writer should believe that the number stated even in this last place is more likely to be correct than any other. Thus a stated result of 2.54 cm (three significant figures) implies that 2.53 or 2.55 are less likely to be correct, and 2.70 would usually be out of the question (though see below). It is permissible to carry one non-significant figure through calculations to avoid rounding errors, but this should be dropped in the final answer. Normally the number of significant figures in a result cannot exceed the number of significant figures in any of the measurements that were combined to give the result.

A pure number (integer) is assumed to have an indefinitely large number of significant figures in this context, that is $2 = 2.000000\dots$. Constants such as $\pi (=3.1415927\dots)$ should be used with enough digits so that they do not limit the overall precision.

Absolute error (plus and minus). A more precise and preferable statement of uncertainty (error) in the result is the direct one of attaching a \pm estimate to the value, for example 2.54 ± 0.02 cm. Usually the error estimate itself will have only one significant figure. Note that the term "absolute" is simply a label to distinguish this mode of statement from others. It does not imply that an uncertainty so stated is any more certain or correct. Note that the value and the uncertainty carry the same units (cm. in this example).

When there are sound statistical grounds for doing so, an error estimate can have more than one significant figure and can encroach upon "significant figure" rules, for example 2.54 ± 0.13 cm. However, a statement such as 2.54 ± 0.79 cm is self-contradictory and should be rewritten as 2.5 ± 0.8 cm.

Relative Error (percentage). An equivalent statement of uncertainty can be made by dividing the estimated absolute error by the value and multiplying by 100% to express the estimated uncertainty as a percentage of the value,

$$\text{e.g. } 2.54\text{cm} \pm 0.8\% \quad (= 2.54 \pm .02 \text{ cm.}) \quad \text{i.e. } \%y = 100\% \times \Delta y/y$$

In many experiments the relative error is constant throughout a series of measurements, though the absolute error may change. Expression of uncertainty in % is obviously more convenient in such cases.

OBTAINING THE UNCERTAINTY ESTIMATE

For each of the steps of obtaining a result value there is a matching step for estimating its uncertainty:

Value:	Uncertainty:
data readings	basic uncertainty estimate
averaging data	uncertainty of an average
calculating result	propagation of uncertainty (combining different types of data)

BASIC UNCERTAINTY ESTIMATES (on the data)

Each piece of data must have its own uncertainty estimate recorded as a \pm or \pm % value. The method for deciding on these basic errors depends on the circumstances and method of taking data. If the estimate is the same for a column of repeated readings, it may appear in the column heading. Otherwise it should be appended to each reading.

It is useful to consider three basic sources of error (uncertainty):

- Variation or uncertain definition of the thing being measured. Examples: How many rain drops strike a square foot of ground per second during a storm? What do you mean by the diameter of a not-quite-spherical ball? How long is the top of a not exactly rectangular table with rounded edges?
- Uncertainty in the measurement itself. Every instrument has limitations on its precision. Sometimes the very act of making a measurement affects the property being measured. Example, measuring the length of a rubber band.
- Applicability of the model/theory/equations to the real situation.
This source applies primarily to the calculation step, but is mentioned here for completeness.

The errors from these three sources can usually be grouped into two categories:

- Systematic errors are those that are always in the same direction and tend to accumulate. Examples: calibration error of a voltmeter, readings from a meter stick missing a few millimeters at the beginning, timing with a stopwatch that runs slow or starts only on the next whole second after you push the button.

Source (c) above refers to the subtle, but important type of systematic error: the use of an inappropriate equation for the calculations. For example, one might have used a constant acceleration equation in a situation where the net force was not constant - e.g. a parachute jumper. Less obvious would be the use of $\langle r \rangle^2$ instead of $\langle r^2 \rangle$ in calculating the moment of inertia of a thick ring. [where $\langle \rangle$ is read as "the average value of" .]

As much as possible, systematic errors should be discovered, quantified, and eliminated or adjusted for in calculations. Just taking more data will not reduce a systematic error, though it may help in the discovery of its source. For a "theory mismatch" error the "correction" is usually to find the right theory to match the experimental conditions.

- B) Random errors are equally likely to be above or below the "truth". Technically, "random" should be applied only in those situations in which the deviations from the mean value can be expected to follow the "normal distribution" or "bell curve". However, it is often used to mean any type of uncertainty that is not obviously "systematic". In fact, methods developed for treating random errors are often applied to estimating the consequences of unquantifiable systematic errors. Random errors can be reduced by taking more data (or by using a better measuring device).

METHODS OF ESTIMATING RANDOM UNCERTAINTIES IN MEASUREMENTS

Statistical Error Estimate: When many measurements are made of a given quantity, random errors are best estimated by studying the variations among the measurements. *Note this method is worthless for estimating systematic errors.*

Mean Value. The mean, or average value of a set of measurements of the same quantity is obtained by summing all values and dividing by N, the number of measurements summed.

$$\langle x \rangle = \frac{\sum_{i=1}^N x_i}{N}$$

The difference between a single measured value and the mean value is called the "deviation" of the single measurement.

$$\delta_i = (\text{value})_i - (\text{mean value}) = x_i - \langle x \rangle$$

Standard Deviation or root mean squared deviation σ is the most accepted way of expressing the amount of scatter of data values from their average value. It is defined and may be calculated by

$$\sigma = \sqrt{\frac{\sum_{i=1}^N \delta_i^2}{(N-1)}}$$

Except for using (N-1) on the bottom, instead of N, σ is seen to be simply the square root of the average (mean) squared deviations. Thus it is often termed the "rms" deviation. In practice, if σ is changed significantly by the use of N vs. (N-1), then N is not large enough to validate the use of statistics in the first place.

An alternative calculation, used by most calculators because it can be continually updated as the data is entered, instead of waiting for completion of all the data, is:

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad \text{where } \langle \rangle \text{ is read as "average of"}$$

The square of the standard deviation is known as the "variance". Physicists prefer to use the standard deviation σ because it carries the same units as the data.

As a rule of thumb don't trust statistics if N is less than at least 10, and even for large N be wary of systematic errors, which can affect the mean while passing unnoticed through the deviations.

About 68%, roughly two-thirds of the measurements can be expected to be within one standard deviation σ of the mean value. Over 99% of the data are predicted by statistics to be within 3 standard deviations of the average. Thus when you report a mean value $\pm \sigma$ (or mean $\pm \% \sigma$) you are said to be stating it at a 68% confidence level. Similarly, a statement of value $\pm 3\sigma$ represents a 99% confidence level.

It is common experimental practice to assume that isolated data outside $\pm 3\sigma$ represent mistakes rather than random fluctuations. Such data points, if any, should be discarded and the remaining values re-averaged.

Limitations on statistical treatment of errors. A purely statistical estimate of uncertainty is **invalid in three common cases:**

- a) Presence of uncompensated systematic errors
- b) Number of readings N too small
- c) Inadequate sensitivity of the measuring instrument

The first two points have already been discussed. For (c) one must recall that validity of the entire statistical treatment assumes that deviations both much smaller and larger than the standard deviation σ can be detected. **Therefore, whenever you observe that a calculated σ is not several times the smallest deviation detectable by the measuring instrument, you must find some other way to estimate the uncertainty.**

Non-statistical estimates:

Manufacturer's statement: Particularly common with electrical instruments, the instruction manual might say for example "voltage: 0.1% or one digit". In the lab this type of information should be supplied by the manual or the instructor. On analog meters the statement might be "voltage scales 2%, resistance 5%" and the user would

have to combine this (by methods discussed below) with the uncertainty associated with 'reading' the dial. This is an example in which a systematic error is treated as a random one. If the voltmeter actually reads, say, 1% too high, it probably does so consistently, *but* since you don't know whether it is reading too high or too low, you treat it as a random uncertainty. Of course, if one has the resources to recalibrate the voltmeter, this systematic error can be reduced.

Least count and common sense: When a statistical estimate is invalid, *and in every case as a check*, one must make an estimate of uncertainty based on experience, common sense, or other information. Every analog measuring device has a smallest, directly readable division - its least count. On a typical lab meter stick the least count is 1mm on the metric side and 1/8 inch on the inch side. When the measurement is made properly, (for example with the meter stick on edge so the markings are tightly against the object being measured) the uncertainty is usually some fraction of the least count. One might read the meter stick to the nearest half millimeter, and the recorded data might be $38.65 \pm 0.05\text{cm}$. Most of the time one should attempt to read analog devices to 1/10 of the least count, e.g. $38.67 \pm 0.01\text{cm}$, but recognize this may be an optimistically small uncertainty estimate. [This "readability" uncertainty related to the least count is often called the "intrinsic uncertainty" of the measuring instrument.] If the circumstances are poorer, for example trying to measure the diameter of a basketball with a meter stick, an honest uncertainty estimate may be several times larger than the least count. This is where the "common sense" comes in. (On a digital meter the least count and the reading uncertainty are automatically the last digit and cannot be subdivided.)

The "counting statistics" of radioactivity:

A single radioactive decay is an extremely fast, but very unlikely event. Any particular nucleus may decay now, or later, or much later; and there is nothing you can do to change when it happens. All you can do is observe a large number of nuclei, "count" the number of events, and treat the results statistically. Your sample may have bursts of decays, and periods of fewer decays, or it may have periods of more or less steady decay rate. The larger the sample the steadier the rate will become in terms of % changes. The question then becomes, "How well does my observed number of "counts" represent the "true" average of the sample's decay rate?

The answer is simply that the standard deviation σ of a number of counts N is the square root of N . So, for example, if you have collected 100 counts in a then σ is 10. You could say that the "true" average decay rate has a 68% chance of being between 90 and 110 for the time interval measured. How can you be more accurate? Collect more counts. If you collected 10,000 counts (by observing longer or using a larger sample) then σ becomes 100 instead of 10. At first glance this seems larger, but compared to N of 10,000 the σ is only 1%, where for $N=100$, σ was 10%.

A second complication in radioactivity measurements in the real world is “background.” Even in the absence of the radioactive source your detectors register some “counts” from cosmic rays, natural or artificial radioactivity in the surroundings, and “noise” in the detectors themselves. These, too, are random in behavior, and your data samples the average. Therefore the contribution of the source must always be calculated by subtracting the background rate – observed without the source, but everything else the same.

Calculation of the resulting uncertainty from combining σ 's of total and background counts will be discussed later.

VALUES CARRIED FORWARD

At this point you have measurements of a few basic quantities, for example the length and width of a table. **All** items show a basic least count uncertainty estimate, and **all** show more than one reading (to reduce the chance of a simple mistake). Some may have satisfied the conditions for valid statistical treatment and thus show also an average value and a standard deviation describing the spread of data around that average value.

Standard Deviation of the Mean. Obviously the average of several readings can be expected to be "better" i.e. nearer the truth than a single reading. The "standard deviation of the mean" quantifies this idea.

The standard deviation σ , calculated above, measures the scatter of data values *from* their mean.

A quantity of greater interest, which should be used in the "propagation of uncertainty" for subsequent calculations, is the standard deviation *of* the mean, **S**. It is calculated from $S = \sigma / \sqrt{N}$ (where N is the number of readings of that piece of data) and *in the absence of systematic errors* measures the likelihood of deviation between the mean of your values and the "true" value. That is, the mean (average) value has a 68% chance of being within $\pm S$ of the "truth", just as an individual data reading has a 68% chance of being within $\pm \sigma$ of the average of the set of readings.

Uncertainty of the uncertainty: Because your data set is not infinite, it is also subject to what is termed a "sampling error." That is, there is an uncertainty about the value of the standard deviation.

For ideal "random" data this can be expressed as:
$$\sigma = \sigma \left(1 \pm \frac{1}{\sqrt{2N}} \right)$$

Thus for ten readings σ itself is statistically in doubt by about 25%, and the doubt increases rapidly for smaller N. This is the reason for suggesting that below 10 readings a statistical

treatment is not worth the trouble (*especially since most data cannot be expected to follow a "normal distribution"*).

Uncertainty in the average of a few readings: It is common to take from half to a quarter of the difference between the extreme values of a few readings as a measure of the accuracy of their mean. Here it is particularly important to be on the lookout for mistakes or anomalous readings (such as a lump of gum on the table where you measured it).

Even if all readings are identical, their uncertainty is not zero. One must still attach a reasonable combination of the "readability" error, based on least count and common sense, and the calibration uncertainty of the measuring device. The value of this reasonable combination, which we shall term the "intrinsic error of the measurement", is usually taken as simply the largest of its components: fraction of least count, common sense, or calibration uncertainty. In calculations of the result you use the mean values of each quantity, and in calculating its estimated uncertainty you "propagate" only the larger of the "intrinsic" uncertainty and the computed statistical error S , not both. Combining these two, while theoretically appropriate, is a subtlety not justified by most sets of data.

UNCERTAINTY ESTIMATE FOR THE RESULT

Propagation of errors is the name of the process by which the uncertainties of the various data values are combined to give the uncertainty in the result. Before this step, all discoverable systematic errors should have been corrected. Remaining unknown systematic errors, for example voltmeter accuracy specifications, have been combined with the appropriate estimates of random error to establish the uncertainty of each mean value carried forward.

Now is the time to combine the various data to calculate uncertainty of the final result.

Rules for combining all data error estimates to obtain the estimated uncertainty of the final result are below. They are based on "first order" error analysis, which is easily understood by examining the total derivative of the expression. (see page 20)

I. When independent data are multiplied or divided, the *relative* uncertainty (fractional or %) in the result is the square root of the sum of the squares of the relative uncertainties of the data.

For example, suppose L and W are the length and width of a table, with measured values $L \pm S_L$ and $W \pm S_W$. Suppose you want the Area of the table, $A = LW$.

Define, for shorthand, $\%W = 100\% \times S_W / W$, and similarly for L.

Then the "first cut" propagation is $\%A = \%L + \%W$

and the "better" calculation is
$$\%A = \sqrt{\%L^2 + \%W^2}$$

Corollary. In multiplication or division the number of significant figures in the result cannot exceed the smallest number of significant figures in any factor. (Integers have, of course, as large a number of significant 0's as needed.)

Suppose instead that you want the ratio of the table's length to its width, $R = L/W$. [Note that first order analysis does not distinguish between L/W and $L \times (1/W)$, i.e. $\%W = \%(1/W)$. This is because by the above rule, $\%(1/W) = \%(1) + \%W$, and the integer "1" has 0 uncertainty].

Thus: first cut $\%R = \%L + \%W$

and the "better" value is

$$\%R = \sqrt{\%L^2 + \%W^2}$$

just like the uncertainty propagation for the area A.

II. When independent data are added or subtracted, the absolute uncertainty of the result is the square root of the sum of the squares of the absolute uncertainties of the data. For example, suppose again that L and W are the length and width of a table, with measured values $L \pm S_L$ and $W \pm S_W$ but this time you want the perimeter, $P = 2L + 2W$.

"First cut" error estimate $S_P = S_{2L} + S_{2W}$
{ see IV below for proof that $S_{2L} = 2 S_L$ }

"Better calculation"
$$S_P = \sqrt{S_{2L}^2 + S_{2W}^2} = \sqrt{(2S_L)^2 + (2S_W)^2}$$

Corollary. In addition or subtraction the result need be carried out only through the first column that contains a doubtful figure.

A different situation for the application of the same “propagation” rules is the calculation of the rate of radioactive decay of a source, from data of background and source+background count rates. If N_T is the total count rate, and N_B is the background, then clearly N_{SOURCE} is $N_T - N_B$. As discussed earlier, the uncertainty in N_T is $\sqrt{N_T}$ etc. The two data values are independent, and therefore their uncertainties combine as the square root of the sums of the squares. The result is that $\sigma_S = \sqrt{N_T + N_B}$. Thus if your data was 700 with the source present, and 300 with the source absent, the contribution of the source would be 400 ± 32 , i.e. the difference in the values \pm the square root of the sum of the counted values.

III. When dependent values are multiplied or divided their % uncertainties are added or subtracted, respectively, *without the squares and square root.*

Dependence most often occurs in calculations involving raising some data to a power. For example, if $C = A^2$, i.e. $A \times A$, then $\%C = 2 \%A$.

This is easily remembered by use of the derivative.

$$\text{if } C = A^2 \text{ then } \ln C = 2 \ln A \text{ and } \delta C/C = 2 \delta A/A$$

Noting δA is S_A , and multiplying everything by 100%,

$$\%C = 2 \%A$$

Note the general rule that if a calculation involves raising a measurement to a power, the % uncertainty in the result is the measurement's % uncertainty multiplied by the power.

$$\text{e.g. if } V = (4 \pi / 3) r^3 \text{ then } \%V = 3 \% r ,$$

since 4, π , and 3 contribute no uncertainty.

A simple example of dependent errors in division would be the measurement of a tabletop, and the calculation of the ratio of length to width. Suppose the huge systematic error were made of measuring the length and width in inches, but recording it in centimeters. The length to width ratio would still be correct because the equal percent errors in both numbers would subtract to zero (leaving only the independent random errors to add as the square root of sums of squares).

IV. When dependent values are added or subtracted their absolute uncertainties add or subtract, respectively, again *without benefit of squares and square roots.*

The most common occurrence of such automatically dependent values is the multiplying of a data value by a constant.

For example if $D = 2X$, i.e. $X + X$, then $S_D = 2 S_X$.

Again this can be justified from the derivative $\delta D = 2 \delta X + X \delta 2$, but $\delta 2 = 0$ since 2 is an integer, and δX is what we mean by S_X . Therefore in the perimeter example in section II above, one would evaluate S_P by first calculating $S_{2L} = 2 S_L$, and similarly for W .

Note the general rule that when a calculation requires multiplying a measurement by a number, the absolute uncertainty of the result is the absolute uncertainty of the measurement multiplied by the same number, but its relative or % error is unchanged.

For example, if one side of a square has been measured and the "result" calculated is the perimeter, then since $P = 4 L$, $\Delta P = 4 \Delta L$, and thus $P = 4L \pm 4 \Delta L$.

However, if all four sides have been *measured independently* with equal instruments and care,

$$\Delta P = \sqrt{\Delta L_1^2 + \Delta L_2^2 + \Delta L_3^2 + \Delta L_4^2} = \sqrt{4 \Delta L^2} = 2 \Delta L$$

This is half the uncertainty in the result, but it requires taking 4 times as much data.

In these examples it is obvious that one term of X , in $2X$, or one factor of A , in A^2 , is dependent on the other. Unfortunately, *experimental* interdependence of measurements is tricky and, if suspected, should be discussed with the instructor. Generally you may assume independence of data except for cases similar to the examples above, most of which arise in calculations rather than in measurements.

PROPAGATION OF UNCERTAINTY IN MORE COMPLICATED CALCULATIONS

Many calculations can be broken down into a series of additions or multiplications. For example the density of a cylinder with an axial hole in it is:

$$\rho = M / [\pi L (R_1^2 - R_2^2)].$$

Therefore:

$$\% \rho = \sqrt{[\%M^2 + \%L^2 + \%(R_1^2 - R_2^2)^2]},$$

but $\%(R_1^2 - R_2^2)$ must come from $\Delta(R_1^2 - R_2^2) / (R_1^2 - R_2^2)$, and $\Delta(R_1^2 - R_2^2)$ comes from $\sqrt{\Delta R_1^2 + \Delta R_2^2}$

and ΔR_1^2 is from $\Delta = \%R_1^2 \times R_1^2 / 100$,
and $\%R_1^2 = 2 \%R_1$, etc.

This is all very tiresome, but straightforward.

There are however some calculations, for example those involving trig functions or logs, that cannot be treated by the simple rules presented above. One can develop a treatment for each case modeled on the total differential, but most students choose instead to simply calculate "worst case" uncertainties by recalculating the result substituting, into the calculation, values increased or decreased by their estimated uncertainties in a way to maximize their contributions (for example using +S for numerator values and -S for measurements in the denominator). The estimated uncertainty in the result is then some reasonable fraction of this "worst case."

COMPARISON OF VALUES.

If one quantity is experimentally determined by two different methods, it is meaningful to compare the two results quantitatively. This is most often done by computing a percent difference. Thus if V_1 and V_2 are the values of V measured by methods 1 and 2, then

$$\% \text{ Difference} = 100\% \times \text{Difference} / \text{Average} = 100\% \times \frac{V_1 - V_2}{\frac{1}{2}(V_1 + V_2)}$$

If a quantity has a standard value (a value accepted by the scientific community), it is meaningful to compare it to your result. Again the most common quantitative comparison is the % difference, but when compared to an accepted value the resulting number is dignified by the label "% Error" rather than "% difference."

$$\text{Thus} \quad \% \text{Error} = \frac{|V_{\text{yours}} - V_{\text{std}}|}{V_{\text{std}}} \times 100\%$$

In either above situation if the two values are drastically different, e.g. one twice or ten times the other, the % form of comparison is meaningless and you should review your calculations and units for a mistake. If no mistake can be found, you must settle for admitting that one value is "so many times" the other. But DO NOT use terms such as "900% error" or "40% correct."

What is a "good" or "acceptable" difference? It depends on the experiment. A useful way of looking at this is to consider the difference in the context of the uncertainties of the values being compared. For the purpose of the instructional labs we will consider the difference to be "adequately small," the values to "correspond," and the results to be "in agreement" according to whether their difference is less than the combined uncertainties of the values,

i.e. whether $(A - B)^2 < (\Delta A)^2 + (\Delta B)^2$.

Of course it is not fair to intentionally inflate ΔA and ΔB to achieve this condition.

ANGLES AND TEMPERATURES present special problems in error analysis, not because both are in degrees, but because in many situations the "zero" is arbitrary and, therefore a % error is meaningless.

An "absolute error" in temperature of $\pm 2^\circ$ is meaningful, but to state the situation of $\pm 2^\circ$ at 20°F as a 10% uncertainty, is wrong because 0°F is arbitrary. However, $\pm 2^\circ$ uncertainty on a temperature *change* of 50° would be a meaningful $\pm 4\%$. A single measurement's % uncertainty becomes meaningful only when the context is such that an "absolute" temperature scale, such as $^\circ\text{Kelvin}$, is appropriate. For example a $\pm 2^\circ\text{C}$ uncertainty would be meaningfully a $\pm 0.7\%$ uncertainty at room temperature of about 300°K if one were estimating the average speed of air molecules.

Similarly, an "absolute" angle uncertainty is meaningful, but % uncertainties make sense only when compared to a difference between two angles, or when there is a clearly understood "zero" - such a comparing to a full cycle, 360° , of the motion.

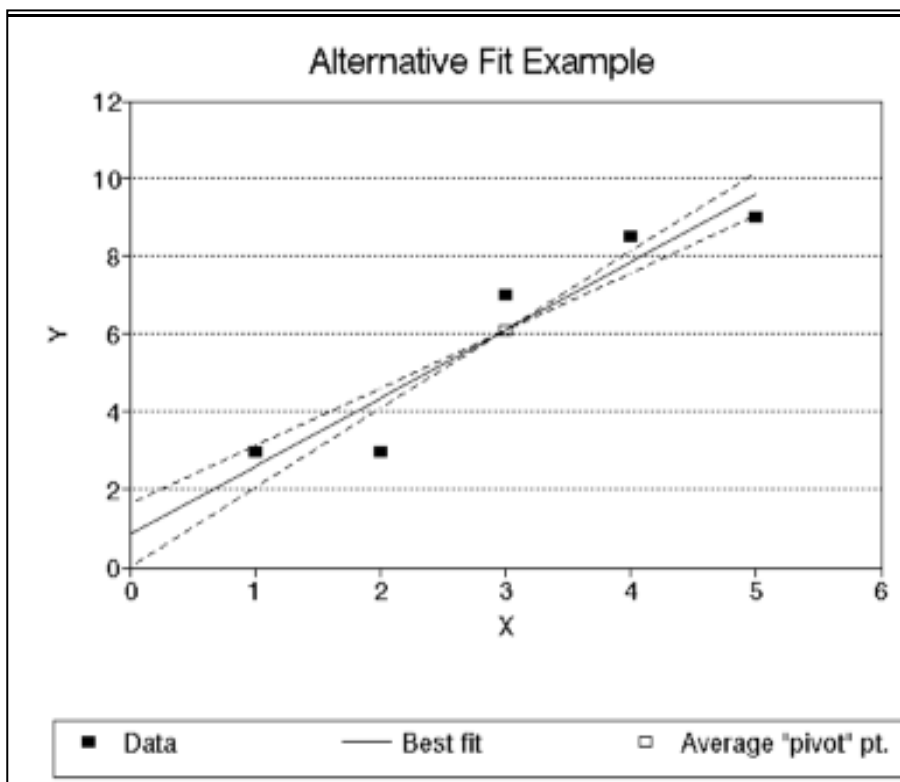
ESTIMATING ERRORS IN GRAPHICALLY OBTAINED VALUES.

Graphical uncertainty estimates:

Elaborate formulae exist for calculating slopes, intercepts, slope uncertainties, and intercept uncertainties directly from data values. Some of these appear below. For lower division physics labs we will, however, generally use an analog approach - PLOT THE GRAPH. Draw what you consider the "best" straight line fit (linear regression) to obtain slopes and intercepts. (This assumes, of course, that the graph axes have been chosen to make the expected relation linear.) Then draw what you see as "alternative, acceptable" fits to estimate reasonable plus and minus limits in your slope and intercept values - referred to in the lab manual as "alternative slope method".

To the right is an example of drawing "alternative fits" to estimate the uncertainty in slopes and intercepts.

(The spreadsheet plot has 'cheated' by using the numerically calculated values for $\pm\sigma$ -slope and forcing the alternative fit lines to go through the average-x, average-y position. The resulting intercept variation is very close to the numerical $\pm\sigma$ -intercept.) It is interesting to note that all lines, especially the higher-slope alternative fit, have a smaller slope than might have been drawn "by eye." As bad as it looks, the coefficient of fit (= correlation coefficient) is 0.946 for this graph!



It is particularly useful for estimating the "alternative fits" to have plotted error bars on your points. In the spirit of standard deviation you can then miss up to a third of the error bars with your "acceptable" extreme lines. [This was not done for the example above.] If even your "best" line cannot touch two thirds of your error bars, you have a problem, [as would be the case above if the plotted "point" size represented the estimated

uncertainty]. You should then restudy the data (for mistakes or systematic errors), the error estimation procedure (for size of error bars) and the theory (for the possibility that the assumed mathematical dependence is in error).

If your pocket calculator does linear regressions, you are encouraged to use it as a check but not as a substitute for your graphical analysis. Calculators have no "sense" about excluding bad data points resulting from data or data entry mistakes, and are therefore prone to producing eight to ten digits of utter nonsense.

Numerical estimates of uncertainty of graph slopes and intercepts

In the GRAPHS section equations are listed for calculating the intercept and slope of a linear graph. Repeated here, these equations [for the $y = a + bx$ line] and n data pairs are:

$$a = \frac{[y][xx] - [x][xy]}{n[xx] - [x][x]} \qquad b = \frac{n[xy] - [x][y]}{n[xx] - [x][x]}$$

where $[y] = \sum y_i$ of all data, $[xx] = \sum x_i^2$ of all data, $[xy] = \sum x_i y_i$ etc. and one is assuming that the x values are accurate and the scatter is in the y values.

In order to estimate the accuracy of the coefficients, one must return to the data-calculation table to calculate the "residuals", i.e. how far each data point is from the best fit straight line.

$$\text{for } d_i = a + bx_i - y_i \quad \text{and } \alpha^2 = [dd]/(N-2) \quad \text{for } N \text{ readings in the data set.}$$

The estimated error in the coefficients is

$$\sigma_a^2 = \frac{[xx] \alpha^2}{N[xx] - [x]^2} \qquad \sigma_b^2 = \frac{N \alpha^2}{N[xx] - [x]^2}$$

and if one is a glutton for punishment and wants the correlation coefficient, one must again return to the table to calculate columns of $X_i = x_i - x_{AV}$ and $Y_i = y_i - y_{AV}$.

$$\text{Then: } r = \frac{[XY]}{\sqrt{([XX][YY])}}$$

Below is a Sample data/calculation Table for numerical calculations of slopes, intercepts and their uncertainties

x	y	theo	xy	xx	X	Y	d	dd	XX	XY	YY
0	4.8	4.41	0	0	-4.5	-9.98	-0.395	0.156	20.25	44.91	99.60
1	6.3	6.71	6.3	1	-3.5	-8.48	0.411	0.169	12.25	29.68	71.91
2	9.5	9.02	19	4	-2.5	-5.28	-0.484	0.234	6.25	13.2	27.88
3	10.8	11.32	32.4	9	-1.5	-3.98	0.522	0.272	2.25	5.97	15.84
4	13.2	13.63	52.8	16	-0.5	-1.58	0.427	0.183	0.25	0.79	2.50
5	16.6	15.93	83	25	0.5	1.82	-0.667	0.445	0.25	0.91	3.31
6	17.9	18.24	107.4	36	1.5	3.12	0.338	0.114	2.25	4.68	9.73
7	20.1	20.54	140.7	49	2.5	5.32	0.444	0.197	6.25	13.3	28.30
8	23.7	22.85	189.6	64	3.5	8.92	-0.851	0.724	12.25	31.22	79.57
9	24.9	25.15	224.1	81	4.5	10.12	0.255	0.065	20.25	45.54	102.41
45	147.8	147.8	855.3	285	0	-0.00	0.000	2.559	82.500	190.20	441.06

intercept $a = 4.405$; $\sigma_a = 0.332$; $\alpha^2 = 0.320$
slope $b = 2.305$; $\sigma_b = 0.062$; coefficient of fit = 0.997

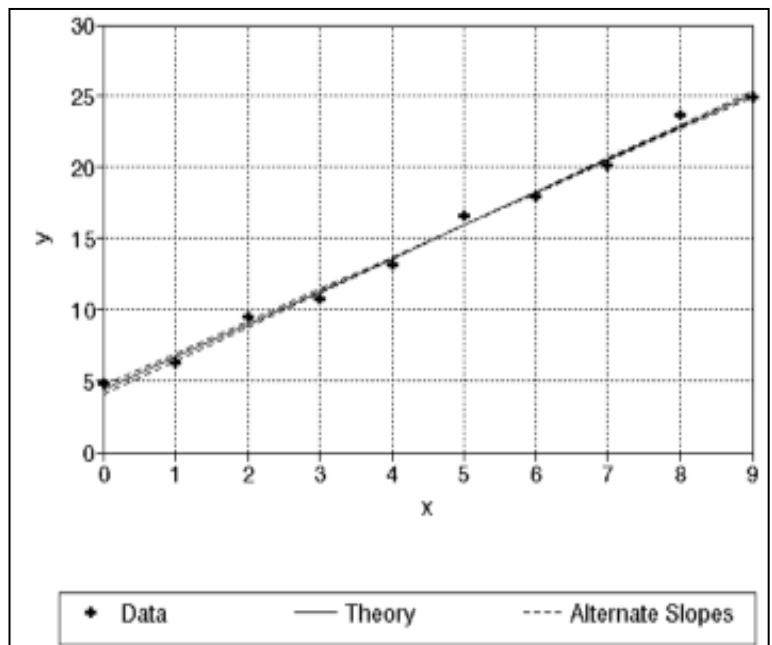
So the equation of the best fit line is:
 $y = (4.405 \pm 0.332) + (2.305 \pm 0.062)x$

and the alternative fit lines plotted are:
 $y = (4.405 - 0.332) + (2.305 + 0.062)x$,
and
 $y = (4.405 + 0.332) + (2.305 - 0.062)x$

Note that these estimated uncertainties are based entirely on the scatter of y values with the x values assumed accurate.

Formulae exist for incorporating uncertainties in the points themselves by weighting them according to their reliability, but these are beyond the scope of these notes.

Reference: J. Topping "Errors of Observation and Their Treatment", Institute for Physics, London 1956



CONCLUSION OF ERRORS SECTION.

When you complete an error analysis and arrive at a predicted uncertainty for a result, pause and ask yourself, "IS THIS REASONABLE?" Check particularly that any statistical treatment of uncertainty has not given an answer at odds with either the intrinsic uncertainty or the observed scatter of data. Consider also possible systematic errors. Graphs are particularly powerful for spotting trends and interdependence of experimental variables.

Additional questions, to be answered in the discussion section of your lab reports, are:

- 1) (always) "Does the stated uncertainty reflect inaccurate measurement of a constant quantity, or accurate measurement of a varying or variable quantity?"
- 2) (usually) "Has the making of the measurement affected the experiment?" [Did your micrometer squash the sample? Did your voltmeter "load" the circuit? ...]
- 3) (sometimes) "Is the precision limited by resetability, or by readability?" [for example by deciding when the cross-hair is on target, or by the subsequent reading of the cross-hair's position on the scale of the setting screws]

DERIVATION OF FIRST ORDER ERROR ANALYSIS

A typical calculus text shows that the “total differential” of a function of several variables, say $w(x,y,z,\dots)$ is:

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz + \dots$$

In experimental measurement terms this translates as: “If w is the calculated result from data measurements x, y, z, \dots , then the change (uncertainty) in this result due to a change (uncertainty) in any of the data contributing to the result can be calculated from the data change times the partial derivative of the combining ‘result’ equation with respect to that data ‘variable’.”

For example, if w is the perimeter of a rectangle, $w = 2x + 2y$, where x and y are the measured length and width.

thus $dw = 2 dx + 2 dy$, and if there is an error or uncertainty in x of dx , then the resulting error or uncertainty in w is $2 dx$. If dx is known to be positive, then so is dw ; and if dx is negative (i.e. too small) when w is reduced by twice the dx error. And if dx is a \pm uncertainty, then dw is $\pm 2dx$. Same for dy .

But suppose *both* dx and dy are non-zero. Here’s where additional thought is needed. If dx and dy are “systematic errors,” as defined on page 5, then they simply add or subtract according to their signs and the sign of their corresponding partial derivatives.

But suppose both are “random” and “independent,” equally likely to be $-$ or $+$. Think of “independence” graphically as “at right angles.”

The most probable combination of “independent” quantities is to take the square root of the sum of squares, analogous to the hypotenuse in two dimension. But of course this process is not limited mathematically to two, or even three dimensions. So for random, independent errors the total differential becomes a mnemonic for the process with the simple summation of partial derivative terms replaced by the square root of the sum of their squares. Note that it is the SUM of the squares of the terms. No uncertainty can ever reduce the uncertainty of the result, even though a known increase, or decrease, in the corresponding piece of data may decrease the result.

For a second example, suppose w represents the density (mass/volume) of an object. $w = m/xyz$ for a rectangular brick of dimensions xyz .

then

$$dw = \frac{dm}{xyz} + \frac{m}{yz} \left(\frac{-dx}{x^2} \right) + \frac{m}{xz} \left(\frac{-dy}{y^2} \right) + \dots$$

and if we divide both sides by $w = m/xyz$, we get

$$\frac{dw}{w} = \frac{dm}{m} - \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} \quad (20)$$

which we could have reached more directly by taking natural logs of both sides of the equation before doing the partial derivatives, i.e.

$$\ln w = \ln m - \ln x - \ln y - \ln z$$

Again, if dx or dy or dz is known to be positive, then it reduces w , just as a positive dm increases w . But if all are independent uncertainties, then the fairest combination is the square root of the sum of the squares of the terms.

Putting equation (20) in words, we state that for a result obtained by multiplying or dividing the data, the “relative error” in w is the square root of the sum of the squares of the relative uncertainties of the independent contributing data.

And if we multiply both sides of eq#20 by 100%, the % error in w is equal to the square root of the sum of squares of the % errors in its components.

Another example: Suppose the object whose density is being measured is a sphere.

$$w = \text{mass}/\text{volume} = m/(4\pi R^3/3)$$

then
$$\frac{dw}{w} = \frac{dm}{m} - 3 \frac{dR}{R}$$

because $4\pi/3$ is a constant with no partial derivative; and the natural log of R^3 is $3 \ln R$. Any mistake or uncertainty in measuring R is tripled. Note that it wouldn't help to measure the diameter D and use $\text{volume} = \pi D^3/6$. The tripled contribution to the error or uncertainty in density is the same.

And what about the minus sign? If R is too big, then w is undervalued; but if dR is a \pm uncertainty, then it increases w 's uncertainty, as always.

And one last example: The period P , the length of time for one full swing, of a simple pendulum is $P = \sqrt{L/g}$ where L is the pendulum length and g is gravity.

Since $\ln(\text{period}) = \frac{1}{2} \ln L - \frac{1}{2} \ln g$, the relative error or uncertainty in period dP/P is only half that of the length measurement dL/L .

NOTE THAT SINCE ALL THE RULES OF PROPAGATION PRESENTED ABOVE ARE DERIVED FROM THE CALCULUS DIFFERENTIAL, THEY ARE VALID ONLY WHEN THE CHANGE OR UNCERTAINTY IS SMALL COMPARED TO THE DATA VALUE.

WHEN A LARGE CHANGE IN DATA IS MADE, IT MUST BE ENTERED FRESH INTO THE APPROPRIATE EQUATION FOR THE RESULT.

OTHER USES OF “ERROR PROPAGATION THINKING”

Clocking your speed:

You are driving down the highway and time yourself against the mileposts. If your speed is 60 mph it takes you $(3600 \text{ sec/hour})/(60 \text{ miles/hour}) = 60 \text{ sec/mile}$. But suppose your time is only 58 seconds.

You could calculate your speed as $(3600 \text{ sec/hour})/(58 \text{ sec/mile}) = 62.07 \text{ mph}$, but this is not easy to do in your head. Instead, remember that $v = d/t$ and therefore $\delta v/v = \delta d/d - \delta t/t$. Assuming the mileposts are accurate, i.e. $\delta d = 0$, then $\delta v/v = -\delta t/t$; $\delta v = -v (\delta t/t) = -60 (-2/60) = +2$ i.e. $v = 62 \text{ mph}$.

QUESTION: Suppose your mile time was 40 seconds instead of 60 sec., what is your speed? NO, it's not 80mph. This is too far away to apply derivative-based error analysis. Your speed is $3600/40 = 90 \text{ mph}$, so watch for airplane spotters and radar checkpoints. (But 56 sec \rightarrow 64 mph by error analysis; closer to the true 64.3 than you can read on your speedometer. And, of course, each second timing uncertainty becomes 1 mph speed uncertainty – if you are driving 60mph.)

Thermal expansion and the pendulum clock:

Almost all materials get bigger as they get warmer. This is covered quantitatively in the topic of thermal expansion, with $\delta L/L = \alpha \delta T$ (where δT is temperature change, L is length, and α is the “coefficient of thermal expansion” with units of reciprocal temperature.) For real materials α is so small that answers stay within the realm of error analysis calculation for any change that doesn't destroy the sample (though changes in α with temperature can cause a problem for large temperature changes).

Meanwhile the period of a pendulum is $P = \sqrt{L/g}$, where g is gravity.

QUESTION: Suppose a clock with a steel pendulum shaft cools by 6 degrees, how much time does it gain (or lose) per day?

ANSWER: $\delta P/P = \frac{1}{2} \delta L/L - \frac{1}{2} \delta g/g = \frac{1}{2} \alpha \delta T$. (Remember those halves come from the square root, and g isn't changing) The value of α for steel is 13×10^{-6} per degree C. P for a day is $60 \text{ sec/min} \times 60 \text{ min/hour} \times 24 \text{ hours/day} = 86400 \text{ sec/day}$. Therefore $\delta P = 86400 \text{ sec/day} \times 13 \times 10^{-6} \text{ per deg} \times (-6) \text{ degrees} = -6.7 \text{ sec}$. The period is shorter; the pendulum swings more often; the clock "gains" 6.7 seconds per day by being cooler by 6 deg. (Note that it doesn't matter whether the pendulum swings once per second, or is so long that it swings only once per day, the % change and therefore the change per day is the same.)

Pendulum clocks on mountains:

Let's carry our pendulum clock 1000m in altitude up a mountain. Outside a spherical mass $g = GM/R^2$ where G is {Newton's} universal gravitational constant, M is the mass (here, of the earth), and R is the radius from the center (6400km). Thus $\delta g/g = -2 \delta R/R = -2 \times 1000\text{m}/6400\text{km} = -3.125 \times 10^{-4}$ (where the "2" came from the squared R). Therefore, remembering that $\delta P/P = -\frac{1}{2} \delta g/g$ we find that $\delta P = 86400 \times \frac{1}{2} \times 3.125 \times 10^{-4} = 13.2$ seconds per day. δP is positive, the period is longer, the pendulum swings fewer times per day, and the clock "loses" 13.2 seconds per day.

QUESTION: You know that it generally gets cooler as you go to higher altitudes. A typical value is 6°C per 1000m up. So what coefficient of expansion is needed for our pendulum shaft material so that the cooling cancels the decrease in gravity? (Be glad your digital wristwatch doesn't care about g , and has a much smaller temperature dependence than a typical pendulum clock.)

SOME QUESTIONS FOR SELF-TESTING

1. If a value and uncertainty were stated as $2.48 \pm .02$ cm, would this error statement be called "absolute" or "relative"?
2. State the value and uncertainty of question #1 using a percent uncertainty statement.
3. What is the mean, standard deviation of the data, and standard deviation of the mean for the following: 2.50, 2.45, 2.38, 2.42, 2.71, 2.65, 2.58, 2.41, 3.68, 2.55, 2.68, 2.51, 2.49, 2.45.
4. You want the area of the lab table top and have recorded the following data for width: 42.35 cm, 42.30 cm, 42.33 cm. What do you do next?
5. Using a vernier caliper you have the following data for length of a brass cylinder:
4.220, 4.220, 4.220, 4.220 cm
 - a) What is the standard deviation?
 - b) What uncertainty should you attach to the mean?
6. Answers from two ways of doing the experiment are 42.1 cm/sec and 265 cm/sec. What is the percent difference?
7. The diameter of a right circular cylinder is $2.78 \pm .04$ cm; its length is $4.2 \pm .05$ cm. What is its volume?
8. Your Geiger counter registers 600 counts in 5 minutes with the sample present, and 400 counts in 5 minutes with the sample removed (so-called "background").
 - a) How many counts per minute are coming from the sample?
 - b) How can I reduce the uncertainty of my answer?

ANSWERS TO THE SELF TEST

1. "Absolute". But remember that this name is a label to distinguish it from "relative", not a guarantee of infallibility.
2. $2.48\text{cm} \pm 0.8\%$. Note that only one significant figure is carried in the uncertainty. If it were a small number, two places might be justified. It depends on how well the uncertainty is known.
3. Mean 2.522;
 $\sigma = 0.106$ or 0.102 depending on whether $N-1$ or N is used. $S = .03$
The 3.68 should be discarded as clearly a mistake or because it is more than 3σ from the mean. Thus the value and uncertainty as an answer, or to be carried into other calculations, would be $2.52 \pm .03$. However, note that while this is fine as a numerical exercise, it would be very unusual to have the spread of data so large compared to the claimed precision of the measuring instrument -- rather like measuring a marshmallow with a micrometer.
4. Start again! This time read the cm side of the stick instead of the inches. And don't try to get by with changing inches to cm by multiplying by 2.54, because the ".3" recorded was probably eighths, not tenths! **ALWAYS LOOK AT YOUR DATA TO SEE WHETHER IT IS REASONABLE.**
5. Standard deviation is meaningless here. You have decided that the vernier line that "lined up" with a main scale line was the one giving a reading of 4.22. If you used the simple 10:9 vernier [where the vernier has 10 marks in the length of nine main-scale marks], then you decided that the lineup was at 4.22, not 4.21 or 4.23, and it must have looked consistently to you to be greater than 4.21 and less than 4.23. Therefore, attach $\pm .005\text{cm}$ (half the effective 'least count'). If you had one of the newer 20:38 vernier calipers, this data represents a decision that the reading was greater than 4.215 and less than 4.225, therefore attach $\pm .005\text{cm}$ (go for the 'least count', not half, because the eye's resolution is nominally .01cm. If you had the latest calipers with a 50:49 vernier, the decision would be among 4.218, 4.220, and 4.222; but the uncertainty would still be $\pm .005\text{cm}$ (eye-and-parallax-limited, unless you use a magnifying glass to justify ± 0.002). Note that in any case you can't take four hundred more 4.22 readings and decrease the uncertainty by a factor of 10, as you could for a "normal distribution" set of data, because it is fundamentally eye-resolution limited, not instrument limited. To gain precision one must enlarge the scale optically (magnifying glass), mechanically (micrometer), or electronically.
6. Percent difference is meaningless for answers so far apart. Look for a mistake. Here, for example, the ratio of the numbers is close to 2π . If this were a circular motion experiment this could pinpoint a likely calculation mistake.

7. $25.5 \text{ cm}^3 \pm 3\%$ or $25.5 \pm 0.8 \text{ cm}^3$.

8. a) $40 \pm 6 \text{ cpm}$ b) count longer

a) In 5 minutes the source contributed $(600 \pm \sqrt{600}) - (400 \pm \sqrt{400})$
 $= 200 \pm \sqrt{1000} = 200 \pm 32$. To express this in counts per minute
both the value and the uncertainty are divided by the 5.

b) Assuming you are studying a constant (long half-life) source,
collecting four times as much data, i.e. 20minute counts each, would
reduce the uncertainty by half. Fourteen hours each of data with
and without source could reduce the statistical uncertainty to ± 0.6 .

X-Y GRAPHS & CHARTS are used for two distinct purposes in scientific reports:

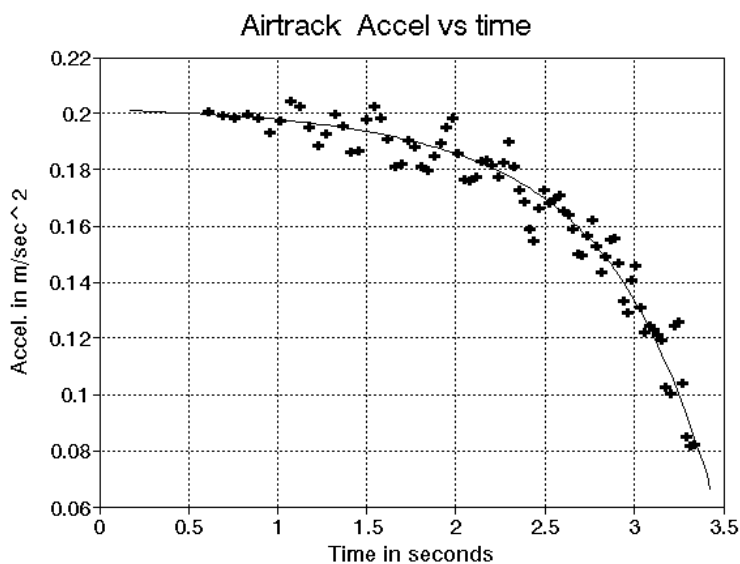
- a) display or presentation of data and results
- b) as an aid to analyzing data and finding mathematical relations among the variables of an experiment.

In both types of use a graph displays as a two-dimensional, or sometimes pseudo three-dimensional picture what happens to one measurable quantity (variable) or predicted result in an experiment as another variable is changed. Graphs are particularly useful because they build on the human mind's innate ability to visually recognize patterns.

The majority of the following discussion is directed toward purpose (b) above, that of graphing as a tool for data analysis and for converting a perceived pattern into a numerical relationship.

PLOTTING

The most common type of graph in physics reports is the "XY" plot. By convention, if one quantity can be said to depend on the other, the dependent variable is plotted on the vertical (ordinate) "y" axis, and the independent variable is along the horizontal (abscissa) "x" axis. That is, "what you do" along 'x' and "what happens" along 'y'. Each point \odot on a graph represents the simultaneous values of the two coordinates (x and y) for one piece of data, where the " · " shows the actual value of the (x, y) pair. Especially for hand-plotted graphs where the number of data points is relatively small, it is common to draw a circle around the data points to make them easier for the reader to spot the dot. The graph as a whole then displays a set of data for which a general relation is sought. [The data graphed here is for a less common situation of accelerating a mass on an air track - using a string mounted higher than in line with the track.]



If several sets of data are to be displayed on the same graph, other symbols commonly used are: \times or $+$, or a dot surrounded with a Δ or \square .

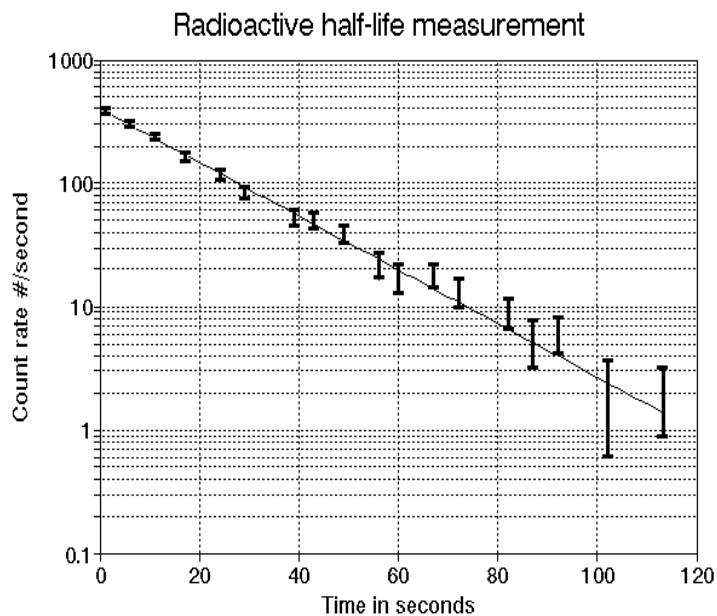
It is sometimes desirable to indicate directly on the graph the uncertainty associated with each data point. This is done with "**error bars**", replacing the " \odot " or "x" or "+" representing the value pair with a \perp , \lrcorner , or \lceil symbol whose height shows the estimated \pm "y" uncertainty of the point plotted, as in the graph at the right.

The scale of the graph is chosen to accomplish the following:

1. Ease of plotting and re-reading. Division of a major unit into 2, 5, or 10 parts is good; 4 is less desirable; other fractions are very awkward. {Note that lines are labeled - not spaces, as in a typical bar graph.}

2. Accuracy. To display the full accuracy of the data the last significant figure should be represented by at least 1/5 of the smallest division on the paper.

3. Size. Data points should occupy as much as possible of the range of both coordinates. The intersection of the axes need not be at (0,0) if there is no data point there (unless graphical evaluation of the intercepts is required). Similarly, one should plan ahead if a negative intercept is expected.



All graphs should be titled and should have their axes clearly labeled - including units. [Try to make the title more informative than simple reiteration of the axes.]

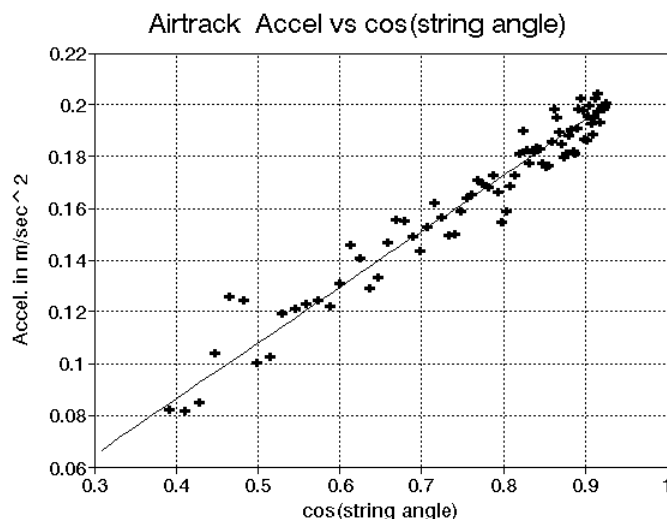
For instructional physics lab reports data must be in ink, but plotting in pencil is recommended.

For most physics experiments a smooth curve is drawn among the points as a first step toward finding a mathematical relation between the dependent and independent variable: viz. $y = f(x)$. The smooth curve need not pass through all (or any) points, but should be drawn to minimize the sum of the distances of all points from the curve. If "**error bars**" are used on the points, the curve should pass through most of them.

The smooth curve should have a shape consistent with the theoretical relation between the quantities.

It is easy to recognize and draw a straight line. It is not so easy to recognize, just by looking at it, a “good” parabola or trigonometric function. **Therefore, whenever possible, x and y should be defined in such a way that this relation will be represented by a straight line. Then, instead of a smooth curve, a "best straight line" should be drawn with a straight-edge.**

{The example at the right is the same data as that in the first graph of page 27, but plotted in a way that “should be” a straight line.} Theoretically, "best line" means positioning the line so as to minimize the sum of the squares of the "y" distances of the data points from the straight line. It can be shown mathematically that the best straight line must pass through the point ($\langle x \rangle$, $\langle y \rangle$), where $\langle x \rangle$ is the average value of all x values of all data points, etc. Therefore, a common aid to drawing a best fit line by eye is to locate the point $\langle x \rangle$, $\langle y \rangle$ on the graph and to treat it as a "pivot point" for the ruler while choosing the best slope.



In instructional physics labs the position of the best line is usually pretty obvious. If the point scatter is so large that the position is seriously in doubt, a data or analysis error probably has been made.

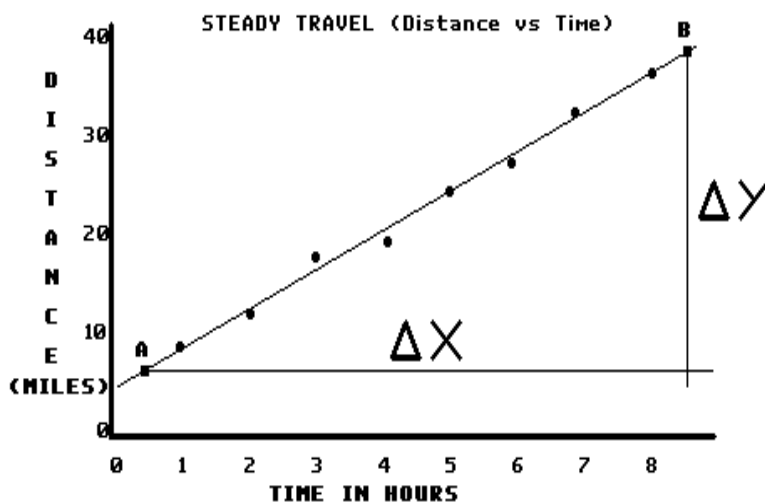
ANALYZING

The goal of graphical analysis is to transform the qualitative, visual relation between the variables into a precise, mathematical one. This is where the value of choosing axes to make the relationship “linear”, i.e. a straight line, becomes apparent. Obviously, “theoretical” guidance is necessary. More about that later.

SLOPE: When the curve is a straight line, as in the example at the right, it follows a mathematical relation of the form $y = a + bx$.

Its slope, defined as $\Delta y / \Delta x$, is the value of the constant "b".
[Its "y-intercept", at $x = 0$, gives the value of 'a'.]

To calculate the slope of a straight line

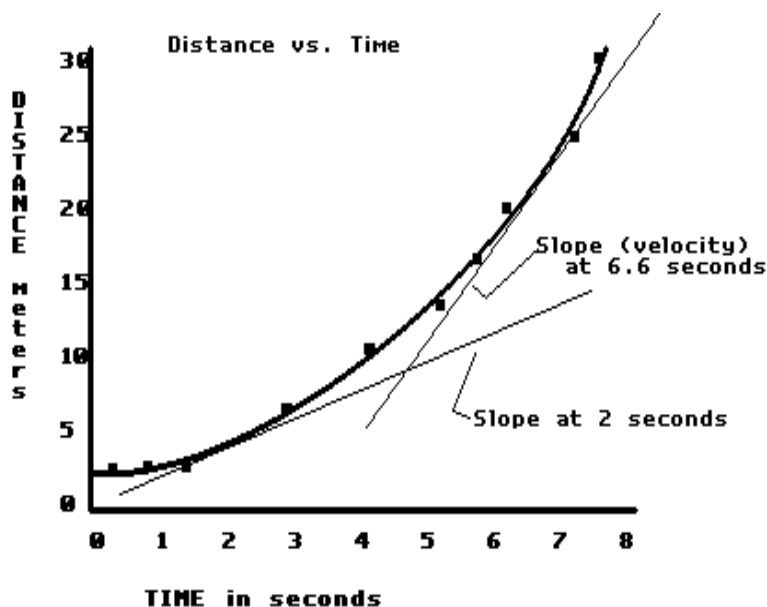


one chooses two points, say A and B, on the line. If the coordinates of A are (x_A, y_A) and of B are (x_B, y_B) , then $\Delta y = (y_B - y_A)$ and $\Delta x = (x_B - x_A)$. The slope is the ratio $\Delta y / \Delta x$.

Clearly the actual slope value of a straight line does not depend on where or how large an interval between A and B is taken. Greater calculation accuracy is possible when A and B are far apart. Points A and B should be chosen on the line, since the line averages all the data and is thus in principle better than any individual points. Therefore, the points from which the slope is calculated usually should **not** be data points.

The units of y and x are preserved when calculating and expressing the slope $\Delta y / \Delta x$.

The slope at any point of a curving line is defined as the slope of a straight line that is tangent to the curve at that point. This slope is different for different points on a curve.



INTERCEPTS: The value of y when $x = 0$ is known as the y -intercept. Similarly the intersection of the curve or its extension with the $y = 0$ line (usually the x axis) is the x -intercept.

Again, if the graph is a straight line, which can be represented by the equation $y = a + bx$, then the y intercept (the value of y when $x = 0$) is the value of 'a'.

Particular care must be taken in evaluating intercepts if the graph has been drawn using other than the line $y = 0$ and the line $x = 0$ as the x and y axes respectively. That is, if $(0,0)$ doesn't show on the paper. [Usually 'a' is then found numerically from $a = y - bx$, picking a point on the line for x and y , and using the slope for 'b'. The x -intercept is $-a/b$]

Hand calculators which do linear regressions may be used as a check on the slope and intercept values, **but not as a substitute for plotting a graph**. Calculators have no common sense about ignoring obviously bad points from bad data or faulty data entry.

INTERPOLATION AND EXTRAPOLATION: Interpolation is the estimation of the value of y for an x value lying between original data points. Drawing the smooth curve among the data values and reading values from that smooth line is an example of interpolation. One extrapolates to obtain the value of y at an x position outside the range of all data. Obtaining the intercepts is usually done by extrapolation.

See the "ERRORS" section of this booklet for graphical estimation of errors in slopes and intercepts and for numerical calculation procedures.

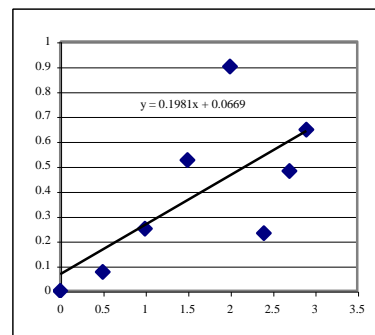
**EXAMPLE I. When does a “best fit,” fit?
(a cautionary tale about not quite constant acceleration):**

While Adam is out of the room getting some water to drink, his partners, Bianca and Charles, take the data shown to the right of position s and time t of a rider that is accelerated on a low-friction track by a string running over a pulley and attached to a hanging weight. They present it to Adam as “Your turn to analyze it.” Adam quickly enters the numbers into his calculator and gets the “best fit” as a straight line $s = 0.0669 + 0.1981 t$.

t in sec	s in meters
0	0.000
0.50	0.076
1.00	0.250
1.50	0.524
2.00	0.900
2.40	0.232
2.70	0.481
2.90	0.647

“There. Four places. That should be accurate enough,” Adam says to himself. He asks his instructor if he can leave lab early and finish the write-up at home.

The instructor says, “No!” and wants to see a plot of the data. Grumbling all the while, Adam plots the data and “best fit” and gets:

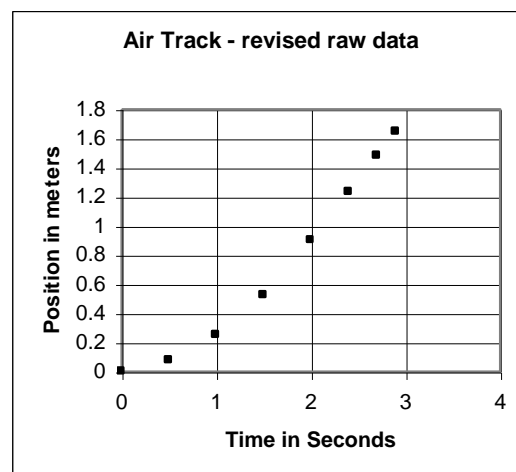


Oops. Adam realizes that he shouldn’t have skipped making the crude plot requested in the lab manual to check for gross data errors.

Presented with this evidence of their carelessness, his partners admit that they failed to note when they were in the ‘second meter’ of the track measurement, and revise the data accordingly.

t	s	s rev,
0	0.000	0.000
0.50	0.076	0.076
1.00	0.250	0.250
1.50	0.524	0.524
2.00	0.900	0.900
2.40	0.232	1.232
2.70	0.481	1.481
2.90	0.647	1.647

Now the crudely plotted curve is smooth, though not straight. (A computer fit might recognize some of this as a piece of a parabola, but by eye it is simply a smooth curve.)

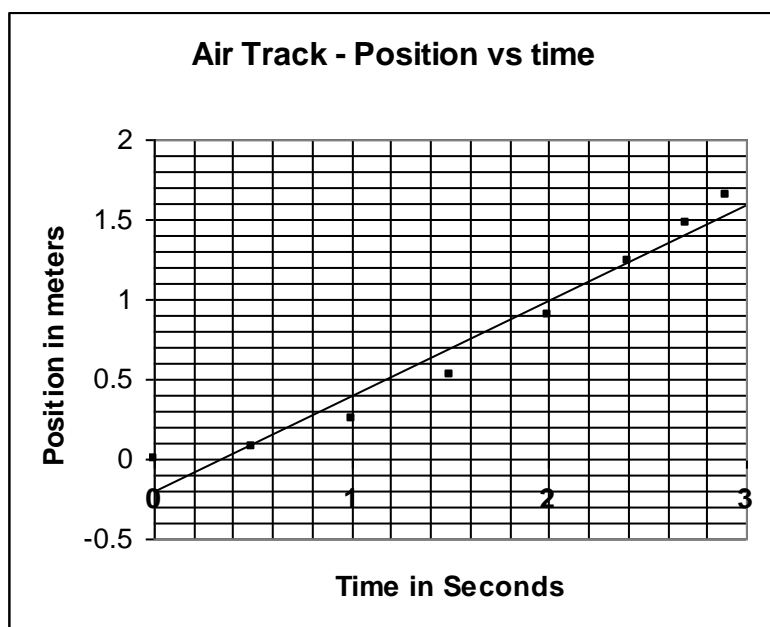


Undeterred by the curvature, which Adam dismisses as another experimental error, he again enters the numbers into his hand calculator and gets a resulting equation for the position of the rider as

$s = -0.206 + 0.5968 t$. “Much better,” he says to himself, and again tries to get permission to leave.

But his lab instructor won’t agree. “I want to see the plot with the best line. And what does your equation mean?” the instructor says.

So Adam plots it more carefully, with the best straight line, and reports, “The starting position was at negative 0.206 and the velocity was 0.5968.”



[Of course his plot is larger and clearer than what shows here.]

“That’s odd,” says his instructor. “I thought I saw the rider accelerating down the track, not moving at constant velocity. And is your data within experimental uncertainty of the fit line?”

To answer this, Adam returns to his partners.

First, they agree that the rider accelerated, and second, their estimate of data accuracy is ± 0.02 sec and ± 0.001 meters (which they should have put on the original data table). Adam

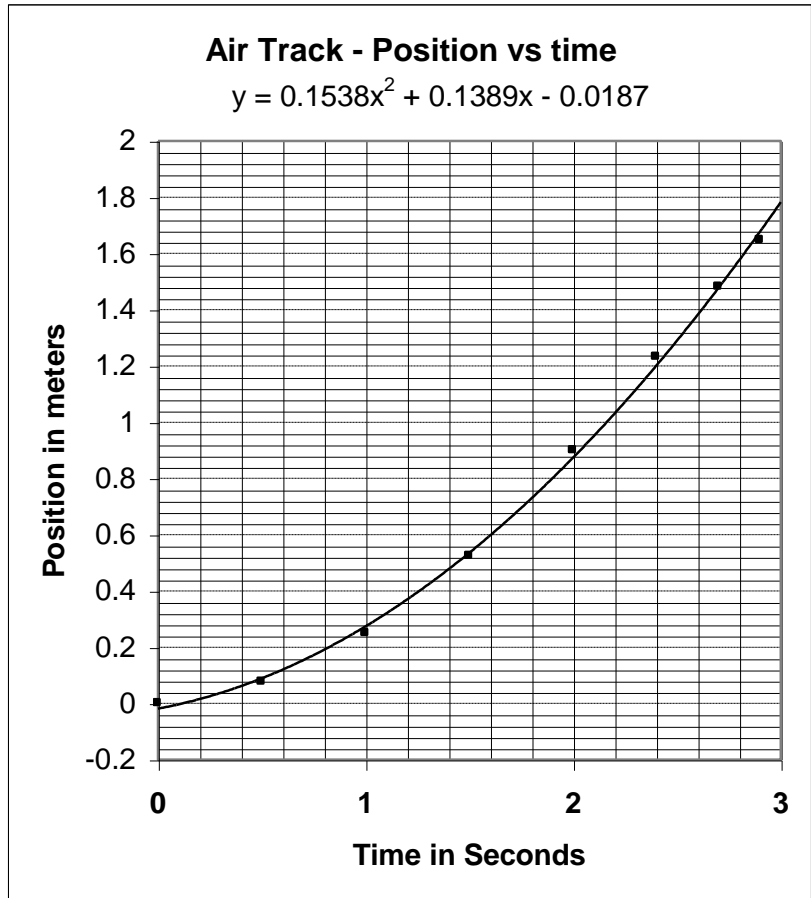
quickly sees that the “best fit” often misses the data by over a hundred times the estimated uncertainty of the points. But this is a small problem compared to the fact that his entire analysis is invalid because, he now realizes, by fitting to a straight line he had incorrectly assumed constant velocity.

Adam (belatedly) begins to think about the experiment. A string over a pulley, attached to a weight should supply a constant force to the rider. Therefore, it is theory for constant acceleration, not constant velocity, on which he should base his analysis. Checking this out by actually calculating a batch of velocities seems like too much work, but he remembers that the kinematics equation for constant acceleration is $x = x_0 + v_0 t + \frac{1}{2} a t^2$. Therefore he enters the data into an available computer spreadsheet, specifies a quadratic fit, and gets the graph shown at the right of the next page.

Adam is happy, but his instructor still isn't. The instructor points out that the computer's "trend line" still misses several of the data points, including the 0,0 point, by nearly 20 times the estimated error in the data.

Therefore, while the line may be a "best fit", it is not a satisfactory fit. Even though this can be a take-home report, the lab instructor encourages Adam strongly to resolve the problem before leaving the lab. Besides, he points out, the manual's instructions specifically request a plot of velocity vs. time.

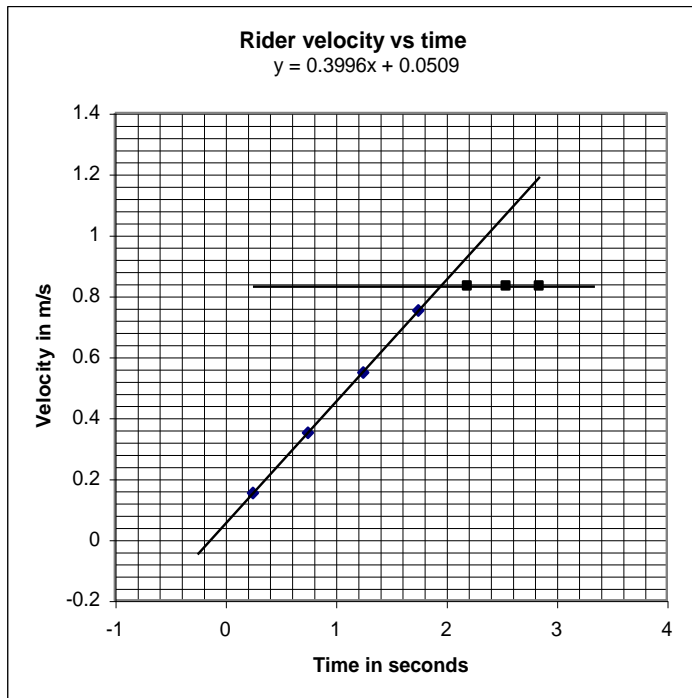
Adam remembers from a homework problem that for constant acceleration the average velocity over a time interval is equal to the instantaneous velocity at the time-center of the interval, so he recopies the data leaving room for the necessary calculations. He plots the results.



t sec	s rev	δs m.	δt sec	t midpt	v m/s
0	0.000				
		.076	0.5	0.25	0.152
0.50	0.076				
		.175	0.5	.75	.350
1.00	0.250				
		.274	.50	1.25	.548
1.50	0.524				
		.376	.50	1.75	.752
2.00	0.900				
		.332	.40	2.20	.830
2.40	1.232				
		.249	.30	2.55	.830
2.70	1.481				
		.166	.20	2.85	.830
2.90	1.647				

Wow!
Look at that.

Two straight lines instead of one overall curve.



No wonder all those attempted shortcuts failed. Adam's plot is, of course, much larger and clearer than what shows here. He can see that all the points are now within a reasonable uncertainty of a line. As a check he numerically evaluates the line's value at a few data times and compares velocities with the table values.

But what happened at the time of 1.95 seconds?

Adam looks at his earlier plot of position vs. time and sees that a time of 1.95 seconds corresponded to a rider position of 0.83 meters on the track. So he goes to the track used for the data, places the rider at what he estimates to be a position of 0.8 from the arbitrary "0,0" data point, attaches the string, runs it over the pulley, and ... There is the accelerating weight just hitting the floor, ending the acceleration. [Clearly something his partners should have noted in the data.]

Finally the lab instructor is willing to sign Adam's data and, since the lab period is over anyway, let him finish at home. He reminds Adam to calculate the acceleration and initial velocity from the velocity graph and to decide how much to round off, for "significant figures," any computer values of the equation's coefficients.

What should Adam learn from his experience?

1. Be part of taking the data.
2. Record uncertainties as well as values.
3. Note other events – like the weight hitting the floor.
4. Do make that crude, as-you-go plot; preferably as data is being taken.
5. Apply appropriate theory.
6. When you need to get results from a graph, plot it carefully and big enough to see whether the "fit" really *fits*.
7. Don't blindly trust computer output. Just being "digital" or displaying a lot of "places" in a numerical result doesn't make it correct.
8. Don't leave the lab before resolving a data problem.

"Log" PAPERS and SCALES

Semi-log and log-log papers and scales are used quantitatively to show relations between quantities when theory predicts exponential or power law behavior. They are also used qualitatively to display data that extends over a very large range of the variables.

When a number is plotted on a log scale its position represents the log of that number and the bother of looking up many logs is avoided. **Therefore, when plotting on a "log" scale you must use the printed numbers, multiplying successive "decades" by powers of 10.** For example, you cannot arbitrarily change " 1 2 3 4 56..." into "3 4 5 6 78...". Care in plotting is necessary, as the value of intermediate intervals keeps changing.

For ease of reading the graph you should supply the decimal point or powers of 10, so that a typical "x" scale would read:

".1 .2 .3 .4 .6 .8 1 2 3 4 6 8 10 20 30 40 60 ..".

Note that there is no "0" on a log scale (because this would correspond to a position of $-\infty$).

Semi-log paper is useful if the theoretical relation is $H = H_0 e^{bt}$.

Since $\ln H = \ln H_0 + bt$, a straight line with slope "b" will be obtained when $(\ln H)$ is plotted against t.

Evaluating logs graphically: The logs (to the base 10) have already been taken by the paper scale itself. That is, the ratio of a distance on the log paper to the length of a "decade", the 1-to-1 distance, is the \log_{10} of the value of the ratio of the endpoints. This process will be made clear in an example below.

Thus for the semi-log slope:

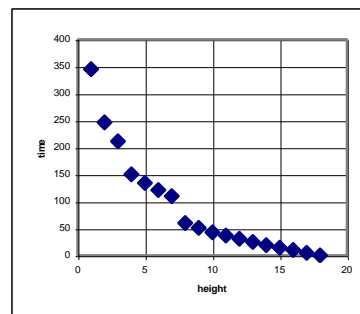
$$\begin{aligned} b &= (\ln H_2 - \ln H_1) / \Delta t \\ &= 2.3 \{ \log H_2 - \log H_1 \} / \Delta t \\ &= 2.3 \log (H_2 / H_1) / \Delta t = 2.3 (D_H / D) / (t_2 - t_1) \end{aligned}$$

where D_H is the distance (in cm.) between points H_2 and H_1 on the line and D is the decade length (also in cm.) "b" must have units so that the total exponent is unit-less, typically sec^{-1} .

Example II, the leaky water bucket.

Adam, Bianca and Charles are still working together. This time they've recorded the times at which the level in a cylindrical water "bucket" passed each centimeter mark on the side of the cylinder.

It's Bianca's turn to lead the analysis. She, of course, begins with a quick, crude plot of the raw data.



Height	time
18	0
17	5
16	10
15	14
14	19
13	25
12	31
11	37
10	43
9	51
8	60
7	110
6	121
5	134
4	150
3	211
2	246
1	345

The step discontinuities immediately indicate that there is a problem. Bianca realizes that as Charles read off the times, responding to Adam's "2, 1, Now's", Charles's "one minute ten" got recorded at "110", etc. She corrects the data accordingly.

Although the data was taken with height as the "independent variable", it will be treated in the analysis as if height is "dependent," on the 'y' axis, with time "independent" horizontally. Therefore, while correcting the data mistake, Bianca also reorders the data columns to treat it as if height is the dependent, 'y', variable.

The corrected data plots as a smooth curve, but it's not obvious how to obtain the numerical relation.

Looking at the "physics" of the situation, Bianca sees that the rate of flow out the leak, and therefore the rate of change of the height, should be proportional to the height.

time	corr. time	Hgt.
0	0	18
5	5	17
10	10	16
14	14	15
19	19	14
25	25	13
31	31	12
37	37	11
43	43	10
51	51	9
60	60	8
10	70	7
12	81	6
13	94	5
15	110	4
21	131	3
24	166	2
34	225	1

$$\delta H / \delta t = b H$$

re-arranging gives

$$\delta H / H = b \delta t$$

which leads to:

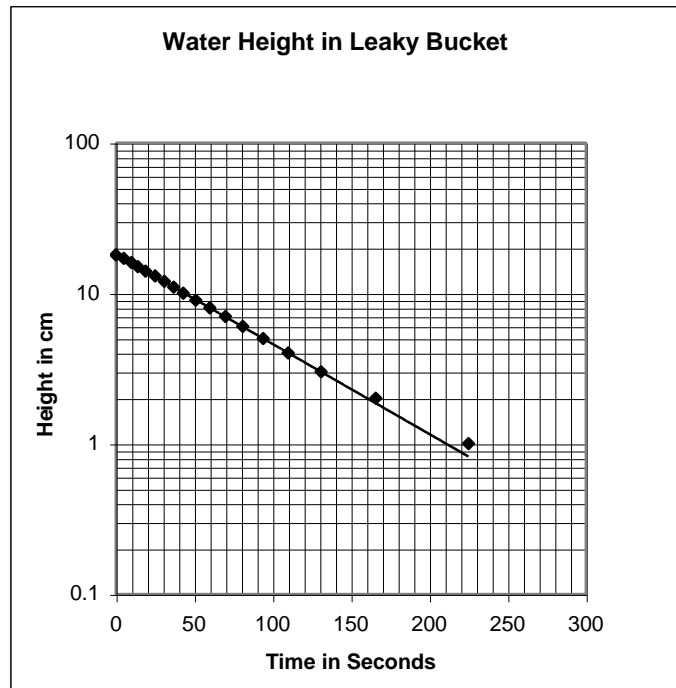
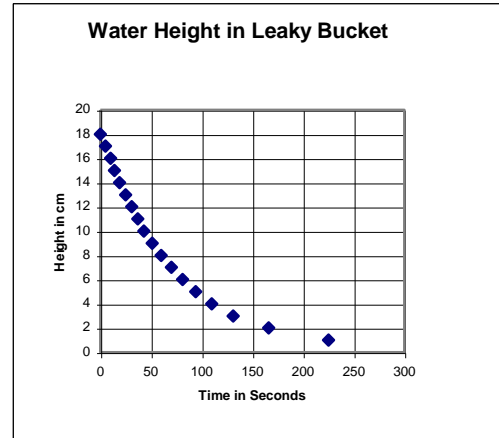
$$H = H_0 e^{bt} \quad \text{or} \quad \ln H = \ln H_0 + bt$$

Therefore, if (ln H) is plotted against t, a straight line of slope b should result. Since H is decreasing with time, b should be negative.

At this point Bianca could laboriously calculate natural logs of all the height values and plot them against time, but she has read the manual. The instructions ask her to use semi-log paper. Since on the semi-log paper Bianca

simply plots the data on the scale provided by the paper, she sees quickly see that her analysis theory is correct. (see next page)

Bianca is initially concerned that the paper seems to be doing log₁₀, while the theory refers to ln, i.e. log_e.



Then she remembers that since

$$\ln x = (\ln 10)(\log x) = 2.303 \log x$$

the shape of the line is not affected. The factor of 2.303 must be remembered in the analysis, however.

Bianca's own graph is, of course, full page size and its additional intermediate lines allow plotting of time and position to approximately the accuracy of the data. In drawing a straight line Bianca sees that she has a choice between a reasonably good fit of all points or an excellent fit of all but the first and the last two points. She chooses the latter. Her reasoning, supported by the instructor, is that the first point was taken under different circumstances (starting the clock while releasing the cork), while others were taken "on the fly." The last two were for a very slow water flow and may have been subject to additional slowing from surface tension or other unknown factors.

The slope, which will provide the numerical value of "b" can be calculated two ways.

method 1: $b = (\ln H_2 - \ln H_1)/(t_2 - t_1)$

taking values off her best fit line at convenient times, Bianca gets

$$(\ln 1.16 - \ln 18.27)/(200 - 0) = (.1484 - 2.905)/200 = -0.0138 \text{ sec}^{-1}$$

method 2: $b = 2.3 (D_H/D)/(t_2 - t_1)$

where D is the size, in centimeters, of a "decade" on the paper,

and D_H is the vertical distance in cm. between points on the line at times t_2 and t_1 .

[Note: the ratio D_H/D gives \log_{10} of the vertical distance between the chosen points.]

By selecting a time interval over which H changed by $\times 10$

(i.e. one decade), Bianca easily gets

$$b = 2.3 (-9.5/9.5)/(210 - 44) = -0.0139 \text{ sec}^{-1}$$

where 9.5cm is the measured vertical length of one "decade" on her graph paper.

As a way of estimating the uncertainty, Bianca also measures the slope of the line she would have drawn if she considered all data of equal value.

She gets $b = -0.0132 \text{ sec}^{-1}$.

Taking this as her "alternative best fit" for uncertainty estimation, on her 'Summary of Results' page she lists the final value of **b** as

$$\mathbf{b = -0.0137 \pm 0.0004 \text{ sec}^{-1}.}$$

[Note that the statement of uncertainty is consistent with the statement of the value, and that both have the same units.]

Log-log paper is used to demonstrate the relation: $I = I_0 r^P$.

Since $(\log I) = (\log I_0) + P (\log r)$, the log-log plot of **I vs. r** is a straight line with slope P and intercept $(\log I_0)$ {read the value of I_0 directly on the printed scale at $r = 1$, (i.e. $\log r = 0$)}.

Evaluation of the slope for log-log paper to get the power P is:

$$P = (\ln I_2 - \ln I_1) / (\ln r_2 - \ln r_1)$$

$$= [2.3 (D_Y/D) / (2.3 D_X/D)] \text{ which is just}$$

$$= D_Y/D_X \text{ \{both measured in cm.\}}$$

i.e. the loglog slope is measured with a ruler, NOT by reading the scale values. *This assumes you are using good quality log-log paper with the same “decade” size (in cm) both directions.*

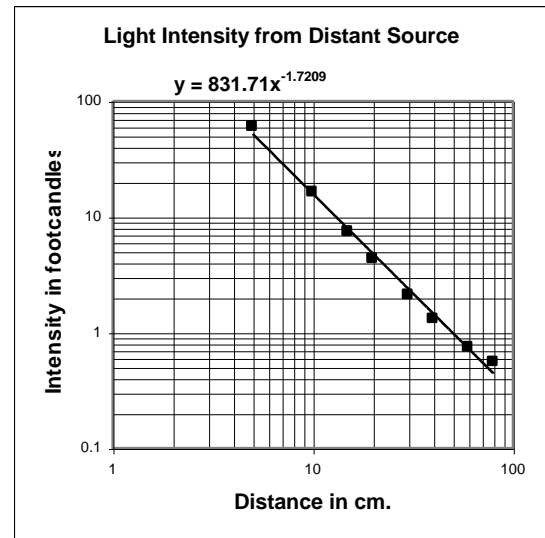
Note that the power P carries no units.

Example: Light intensity vs distance from a small source

The $\alpha \beta \gamma$ team, as they now call themselves – to sound more scientific – is still together. They have made measurements of the light intensity (brightness) at a detector that is moved farther and farther from the light source. It’s Charles’s turn to lead the analysis of the team’s data.

Charles knows that light from a ‘point’ source should obey the “inverse square law,” a power law that can be evaluated using log-log paper.

Never one to “waste time” on hand graphing when a computer is available, Charles enters the data into a spreadsheet, asks it to add a trendline best fit, and prints out the result. He then dutifully measures the slope with a ruler, getting approximately -1. Except for wondering why the computer statement of exponent is about -1.7, compared to his -1, Charles is satisfied.



But Adam and Bianca disagree with Charles immediately.

Distance r cm.	Intensity ft cndls
± 0.2 cm	$\pm 5\%$
5	60
10	16.0
15	7.4
20	4.3
30	2.1
40	1.3
60	.74
80	0.55

“The line misses several points by more than 15%,” says Bianca, “and our light meter claimed 5% accuracy. Therefore, the trend line is not a satisfactory fit”

Adam points out that the computer’s printout of the graph doesn’t have the same decade sizes in y and x. This would explain the discrepancy between slope calculations.

Charles counters that, since the data are not, in fact, in a straight line it will be impossible to get a “satisfactory” fit. The group brainstorms on this problem for a few minutes and concludes:

1. Stray light from the room would make the distant dim values read high. Thus the last three data points should be ignored.
2. At the closest distance, the distance was not “many times” the size of the source. Thus the source might not be acting as a “point” source obeying the inverse square law. Therefore the closest data value also should be ignored.
3. They need to hand-plot on ‘good’ paper so that the “decade boxes” are square.

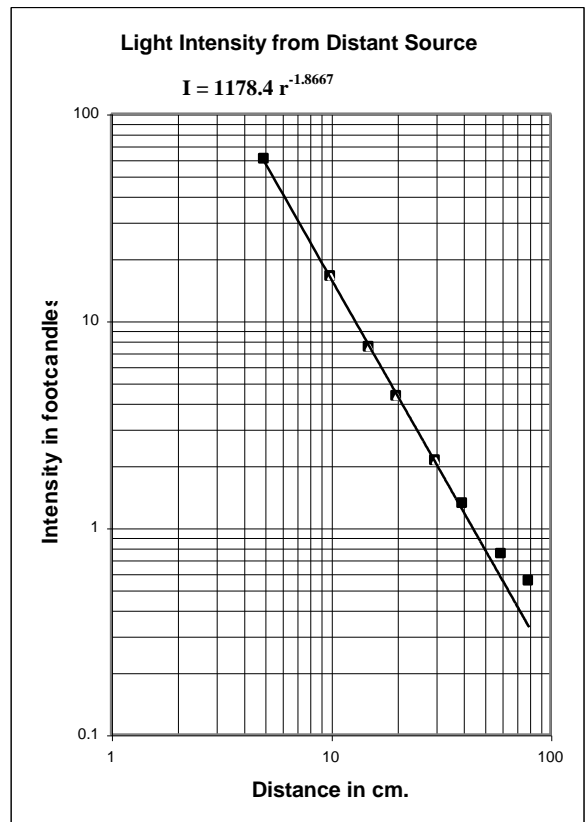
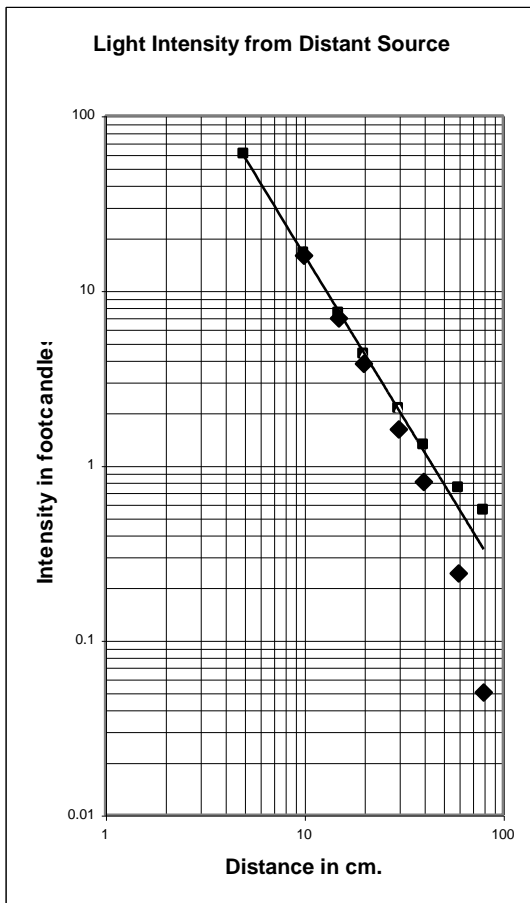
When this is done and they draw a “best fit straight line” ignoring the first, and the last three points, they get a full page version of the graph at the right.

Measurement of the slope, with a ruler on their ‘good’ graph, gives a value for P of -1.90. (The computer now gets -1.867). Adam and Bianca are satisfied, but Charles decides he needs to redeem himself for the initial mistakes – and get some use out of the spreadsheet after all. He suggests, “If stray light made the last points too high, maybe it affected all the values, just more subtly. On the spreadsheet I can easily subtract a fixed value from all the intensity points and see what it does to the fit.”

His first try, subtracting 0.5 ft cndl from all points gives the lower points of the graph at the left.

Clearly an overkill, but it shows what a large effect an added constant value can make to a log-log plot.

Clearly an overkill, but it shows what a large effect an added constant value can make to a log-log plot.



By trial and error, Charles finds that a subtraction of 0.3 foot candles, representing a constant stray light contribution, gives the best fit. The group adds this information to their hand-plotted graph at the right.

Their graphical measurement of the slope now agrees with the computer's trend line (and with expectations of the theory) as $P = -2.00$ [no units].

They decide to state the final result for the power law as:

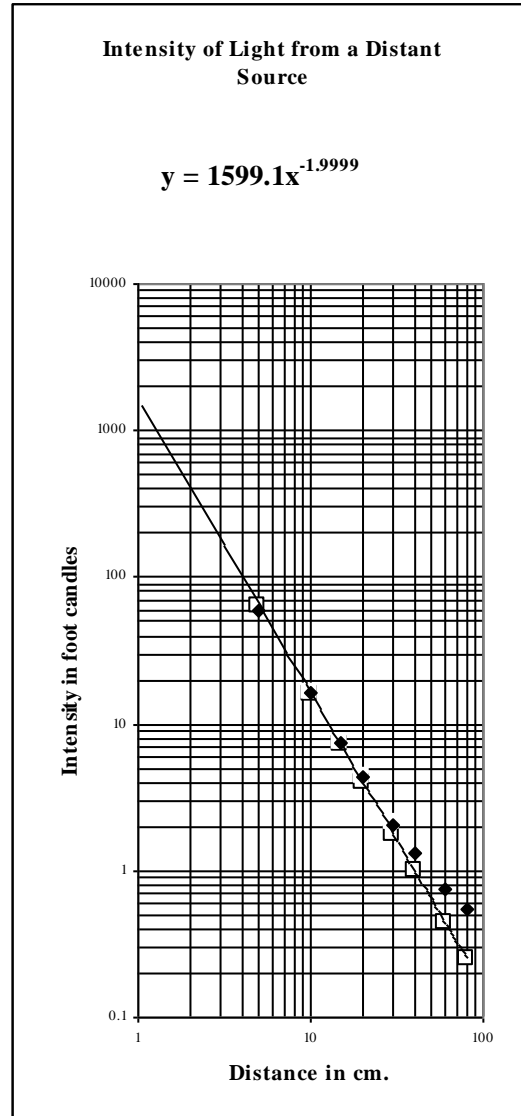
$$P = -2.00 \pm 0.05,$$

with the uncertainty based largely on how quickly the slope changes when differing values of stray light are assumed. That this uncertainty is less than the light meter's specification of 5% does not, and should not concern them. It is reasonable to assume that the light meter's constancy is better than its accuracy - and if it consistently reads, say, 5% too high, this would have no effect on the slope and power law calculation.

Note on log-log paper intercepts: On log paper the distances in cm are proportional to the logs and each decade is a power of 10. Hence 10^{-6} is 6 decades left of $10^0 = 1$, and thus zero = $10^{-\infty}$ is ∞ decades away, far off the paper to the left or bottom. Remember that the equation $I = I_0 r^P$, plots as a straight line only in the form $\log I = \log I_0 + P \log r$. Thus the intercept, giving $\log I_0$ is found where $(\log r) = 0$, i.e where $r = 1$, not where $r = 0$. Therefore, the I_0 value can be read directly on the graphed scale at $r = 1$. [1600 in the example above]

Alternatively, after you have solved for the power P , the constant I_0 can be found by picking a specific point (x_R, y_R) on the experimental line, reading I_R at point R, and calculating I_0 from

$$I_0 = I_R R^{-P}.$$



IN SUMMARY:

Semi-log: to evaluate $H = H_0 e^{bx}$ type relations (plotting log of "y" coord)
 $H_0 = (\text{y value at } x = 0)$ read number off the scale, not distance, don't take log.

exponent "b" = $2.30 (D_Y/D)/\Delta X$
or $(\ln Y_2 - \ln Y_1)/(X_2 - X_1)$ where D's are distances, but X and Y are numerical values.

log-log: to evaluate $I = I_0 x^P$ type relations
 $I_0 = (\text{I value at } x=1)$ read number off the scale, not distance, don't take log.

Power $P = \Delta y/\Delta x$ both measured in distance, **not** numbers on the scale.

Be sure to use 'good' graph paper on which the 'decade' size is the same in both directions.

NUMERICAL CALCULATION OF SLOPES AND INTERCEPTS

Most scientific/engineering calculators now provide linear regression "best fits" for the basic straight line relation. Some calculators, and most of the computers in the labs also provide least squares fits for a few other 2-parameter equations and/or least squares fits for a polynomial of several terms. Often a "coefficient of fit" is given; the most common form of which reports "1.00" for a perfect fit (all points exactly on the line) and down to "0" for no correlation between the two variables. Very few provide a standard error estimate for the coefficients defining the best fit.

The equations which produce these values are as follows
[for the $y = a + bx$ line]:

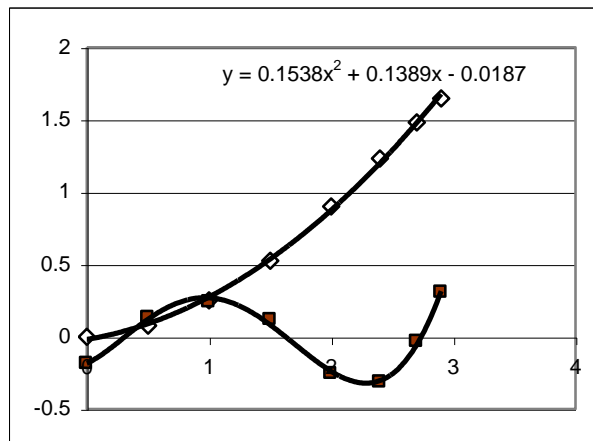
$$a = \frac{[y][xx] - [x][xy]}{n[xx] - [x][x]} \quad b = \frac{n[xy] - [x][y]}{n[xx] - [x][x]}$$

where: n is the number of data x_i, y_i pairs taken,
 $[y] = \sum y_i$ of all data, $[xx] = \sum x_i^2$ of all data,
 $[xy] = \sum x_i y_i$ etc. and one is assuming that the x values are accurate and that the scatter is in the y values.

FOR A NUMERICAL EXAMPLE OF THE USE OF THESE FORMULAE AND BOTH GRAPHICAL AND NUMERICAL ESTIMATES OF THE UNCERTAINTIES IN GRAPHICALLY OBTAINED VALUES, REFER TO THE "ERRORS" SECTION OF THIS BOOKLET.

DIFFERENCE GRAPHS

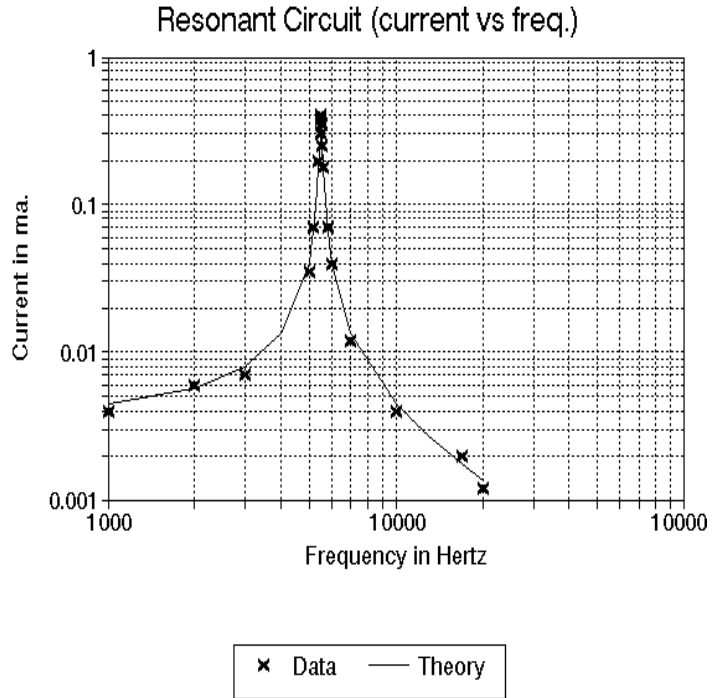
When the accuracy of the data is greater than can be displayed on a moderate sized graph, the "difference plot" becomes a way of magnifying and making visible the difference between the data and the smooth line, or between the data and a theoretical fit. This is particularly useful for computerized plots, where the graph resolution is almost always poor and unreadable. For example, if in "Example I" Adam had done a difference plot between his data and his quadratic fit, he would have had the lower line in the graph at the right. (The "wavy" difference has been magnified by 10 for the plot.) He would have seen immediately that the differences were many times the expected uncertainty, and that they were clearly systematic, not random.



GRAPHING NON-LINEAR RELATIONSHIPS

Graphs are also useful to display relationships between variables even when the relation cannot be reduced easily to a linear one. Often a smooth curve is drawn to guide the eye along the data. **A better treatment is to plot a theoretical graph on the same coordinates as the data**, adjusting the parameters of the theoretical equation to make it fit the data points. Computer spreadsheets are particularly useful for this approach.

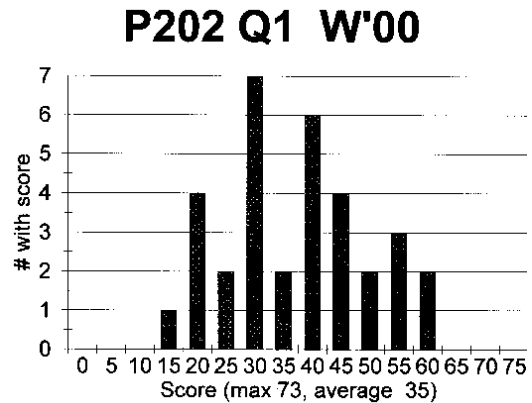
Example: Current vs frequency for an RLC resonant circuit. In this example log scales are used simply to display a huge range of values on a single graph.



FREQUENCY DISTRIBUTIONS:

A less common type of graph used by scientists is the frequency distribution. Here "what you observed" is plotted on the horizontal "x" axis, and "the number of times you observed that value" is plotted on the vertical "y" axis.

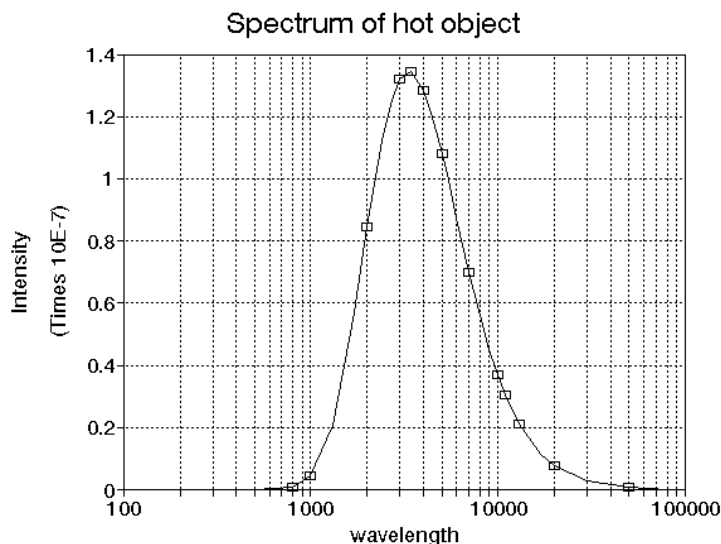
HISTOGRAMS When the "x" values are grouped together so that the graph becomes a vertical-bar graph, the plot is known as a histogram. A typical example is the exam score distribution for a class. Here an "x" label of 10 includes scores from 6 to 10, etc. When plotting a histogram, care must be taken not to "double count" the ends of an interval. For example, do **not** make one bar represent the number of students scoring 10, 11, 12, 13, 14, or 15; and the next 15, 16, ... 20; and so on. This would count those scoring 15 twice.



Another example is the plot of the count rate from a constant radioactive sample (x) and the number of times that rate was observed, when many observations have been made.

Frequency distributions in which the observed quantity should have been a fixed value, but its measurement varied, are the basis of the error analysis of the first section of this booklet.

SPECTRUM When the "x" axis of a histogram is grouped so finely that it becomes effectively a continuous variable, the frequency distribution is known as a spectrum. In a typical spectrum, at the right, the "x" value is the wavelength of light from a hot object and the "how often a photon of that wavelength was observed" becomes the intensity of light "at" that wavelength. In this case the frequency distribution is treated as a normal XY plot.



CALCULATOR LINEAR REGRESSION “BEST FITS” The basic rule in physics instructional labs is that you should not use a calculator result without an accompanying graph that demonstrates that the numerical result really fits the data. Without a graph that includes the plotted best fit line there is no way to check whether the data “makes sense,” or even if a mistake has been made in entering the data into the calculator. *However*, a calculator’s use is perfectly acceptable, even desirable, to check a slope or intercept calculation from a hand-plotted graph.

When using a calculator’s result one must be careful not to confuse “number of places displayed” with “accuracy of result.” The latter must be estimated separately [and is the subject of the first section of this booklet].

COMPUTER PLOTTING (Spreadsheets) The basic rule (default condition) is that computer-generated graphs are acceptable in addition to, but not as a replacement for hand-plotted graphs in the physics instructional labs. An obvious exception is for those labs in which the primary data is taken by the computer and goes directly into a dedicated computer analysis and plotting program.

It is true that modern computer spreadsheets such as Excel or QuatroPro make producing good-looking graphs very easy – perhaps too easy. Instructionally, that is one count against them. There is nothing about keyboard numerical entries or mouse clicks that helps one appreciate how a graph displays the relation between physical quantities, or that teaches the skills of plotting or reading results from a graph. The other instructional deficiency is that the printout of a computer graph rarely has the resolution to verify whether a “best fit” line really fits within the accuracy of the data, or to allow precise reading of interpolated or extrapolated values.

On the positive side, spreadsheets are very useful for examining “what if” situations, such as the effect of stray light in the log-log graph example above, starting page 38. Computer graphs are also handy for getting a quick plot to see whether the data makes sense, or for easily switching between regular and logarithmic plot axes. A spreadsheet also makes it easy to generate a “difference plot,” which displays the difference between the data and the trend line. This can facilitate discovery of systematic errors or mismatches between the experiment and the theory being applied. And, finally, the easily accessible “trend line” on the computer graph is a good check on what has been calculated from the hand plot. (One needs to be careful not to believe all the displayed “places” of the trend line equation, however.)

A FEW QUICK HINTS ON COMPUTER PLOTTING.

- a) **Very important:** one must specify the “x,y (scatter)” type plot for most physics lab graphs. The “line”, “area”, and “stock” types have some options that look superficially like x,y graphs, but these all determine the horizontal position from the data number (1st reading, 2nd reading ...), all equally spaced, rather than from the ‘x’ data value.
- b) To display all the data, but calculate a “trend line” which ignores some points, as was done in several of the above examples, copy the part of the data to be fit into a new column and make it into an additional “series” for the plot. Then make those points themselves small and colorless using the right click “format data series” menu.
- c) If you wish to reformat the display of a particular data series, but are having trouble finding it for ‘right click’ selection, temporarily move one of the data points dramatically to an easily found location. Remember to move it back after the change has been accomplished.
- d) Note that the drop-down menu from “Chart” after a ‘left click’ selects the whole graph, is slightly different from the ones available from clicking, left or right, at various places on the graph.
- e) When using a computer’s trend line, be sure to restrict the fit to something justifiably related to the theory. Higher order polynomial fits will, for example give better coefficients of fit, but produce nonsense extrapolations. Furthermore, the coefficients are no longer easily identified with velocity, acceleration, etc. in simple kinematics situations.

Three Sample problems for your Spreadsheet.

1. The data to the right describes an object found in many beginning physics labs. Describe the object.

horizontal	vertical
8	18
1	4
5	21
3	8
6	14
2	6

2. What is the relation between period (year length) and orbital size for the planets?

size	days in yr.
0.387	88
0.732	225
1.000	365
1.520	686
5.200	4329
9.540	10770

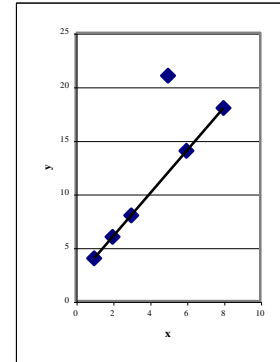
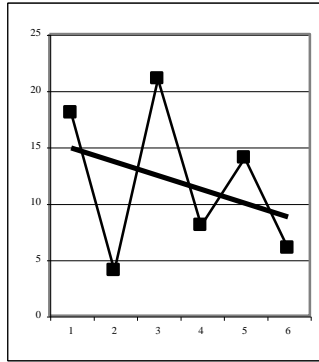
3. A leaky bucket again, but this time your partners measured height from the table top, instead of above the hole. How far is the hole above the table?

cm. height	time
20	1
19	5
18	10
17	14
16	19
15	24.5
14	30
13	37
12	43.5
11	51
10	60
8	80.5
6	110

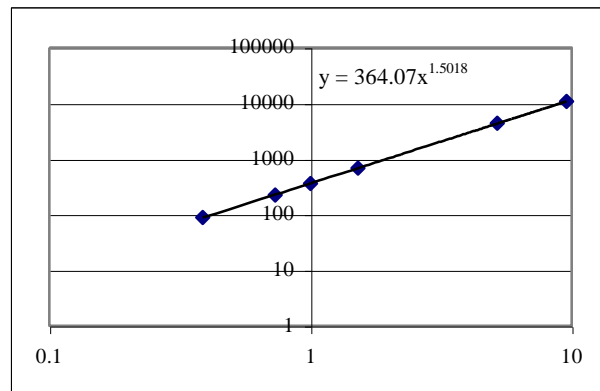
Answers:

1. If you got the graph at the near right, you did a “line” plot, rather than “x,y.”

If, however, you did an x,y plot, and realized that the y value for x=5 was in error (digits transposed), then you identified the object as an inclined plane with a slope of 2 and a starting height, at x=0, of 2.



2. Period proportional to (orbit size)^{3/2}, and Kepler discovered it without calculators, computers, or even log tables!



3. 2.0cm. Unfortunately, the coefficient of fit, R^2 , can't distinguish among 1.8 through 2.2.

”UNITS” ARE IMPORTANT

“A journey of a thousand millimeters begins, and ends, with a single step”

Units are important throughout any scientific measurement, from data to calculated results. The value (number) of a result or a piece of data is nullified if you don't know what it was measured “in.” Furthermore, if the units of a final result are not appropriate for its function, you know it must be wrong. Units have, in effect, veto power over the correctness of an answer – whether it is the answer to a homework problem or the culmination of weeks of research. If the units are wrong, the answer is wrong.

Unfortunately, having the desired units does not prove a result is correct, but it is a good indication that any calculation error is a unit-less one such as a factor of ½, or 4, or pi.

UNIT CONVERSIONS; UNITS ALGEBRA

In “units algebra,” just as in “numbers algebra,” you can always multiply by “one” without changing anything. So, for example:

$$1\text{kph} \left(\frac{1000\text{m}}{1\text{km}} \right) \left(\frac{100\text{cm}}{1\text{m}} \right) \left(\frac{1\text{in}}{2.54000\text{cm}} \right) \left(\frac{1\text{ft}}{12\text{in}} \right) \left(\frac{1\text{mile}}{5280\text{ft}} \right) = 0.62137\text{mph}$$

note that the value of each ratio in parenthesis is exactly one. The extra trailing zeros on the inches/cm bracket are just to emphasize that this, too, is a precise, defined ratio that is good to any number of places.

Many “real life” situations do not require an exact conversion. For example, if you are driving a car with an American-made speedometer in Europe, note that 0.6214 is close to 5/8 and between 0.6 and 2/3. So if the speed limit says 50, drive 30; 80, drive 50; 90, drive under 60; etc.

Some other useful approximations: a liter is about 10% larger than a quart, a meter is about 10% larger than a yard, and 1 mile \cong 1600 meters.

For the more complicated conversions that come up in calculations, keep in mind a few basic equations.

Newton's $F = ma$ gives you as a valid “one” bracket, $\left(\frac{\text{Newton sec}^2}{\text{Kg meter}} \right)$

and Einstein's $E = mc^2$ supplies you with: $\left(\frac{\text{Joule sec}^2}{\text{Kg m}^2} \right)$

In a consistent set of units, e.g. SI, a number one (1) is understood to be in front of each unit word in the above parentheses. When mixing families of units, numerical values appear in the conversions again,

as in $\left(\frac{1 \text{ horsepower}}{746 \text{ Watts}}\right)$ or $\left(\frac{1 \text{ gallon}}{3.7853 \text{ liter}}\right)$ or $\left(\frac{1 \text{ electronvolt}}{1.60 \times 10^{-19} \text{ Joule}}\right)$

Always remember that the argument of a trig function, logarithm, or any exponential must be, as a whole, unit-less. You cannot raise 10 to a meter power, calculate the log of a Joule, or find the sine of a kilogram!

Two units are worthy of special mention: the degree (temperature) and the radian.

Kelvin, Celsius, and Fahrenheit degrees are all equally good for comparing two temperatures to state which is hotter and by how much. But only Kelvin can be used for calculations leading to energy or speed of molecules where, combined with other constants, it is involved in being raised to a power or fractional power.

One revolution = $360^\circ = 2\pi$ radians. Thus degrees of angle and radians are both defined as fractions of “once around” and have no real units to call their own. However, radian is the unit which must be used in dynamic calculations such as $F_{\text{CENTRIPETAL}} = m r \omega^2$ to avoid further conversions. You can take the sine of a radian, or have radians in an exponent, etc. It is wise to carry the radian unit through all calculations, dropping it only at the end if the final answer is not a statement of angle.

SOME HINTS ON TAKING GOOD DATA

ALWAYS KEEP A TIME LINE. For the instructional lab, just the date is usually enough. But for “real world” research a more detailed record of time of data is important. For example, the first firm evidence that “cosmic rays” were of extraterrestrial origin was from correlating count rate to time of day.

DISTANCE MEASUREMENTS:

1. Meter sticks: Use on edge, with the markings touching the object being measured, if possible. Don't trust the stick's ends. If you have to use the end, such as measuring the distance from a wall, first check with a second meter stick that the end of the one you're going to use is neither short nor extended.

If you absolutely can't get the markings next to the object, find some reliable way to avoid parallax (the dependence of the reading on the position of your eye): a drafting 'square', a bubble level, a plumb bob. Even just sighting across the width of the meter stick to use the mm markings themselves is better than nothing. Best is to arrange a mirror behind the object. When you position your eye so as to line up the markings, the object, the reflection of the object, the reflection of the scale markings, and the reflection of your eye pupil, you are viewing exactly perpendicular to the mirror. The mirror must, of course, be in the same plane at both ends of the measurement. Usually this works best using a single flat mirror larger than the object.

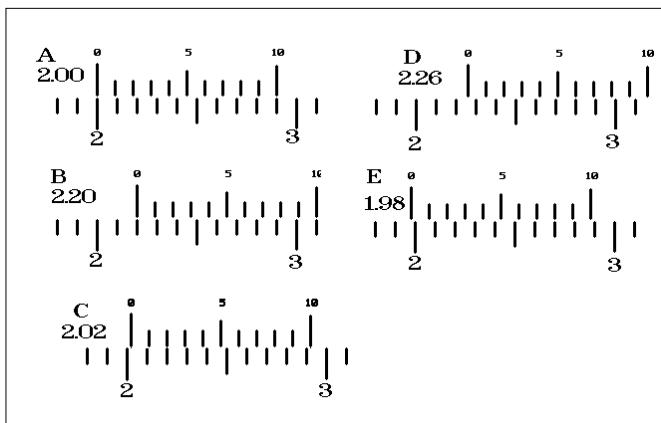
2. Vernier calipers: The function of the vernier is to reliably sub-divide the least count of the main scale. The basic reading is the location of the "0" mark of the vernier against the main scale. Any time the total reading disagrees with what you would have read without interpreting the vernier, you have made a mistake. The vernier only gives you what would have been estimation places without it.

Special care in interpreting a vernier is needed in three cases:

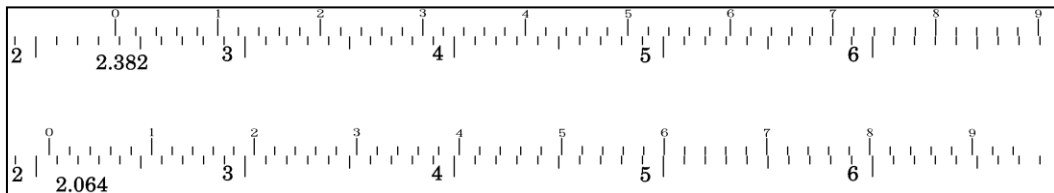
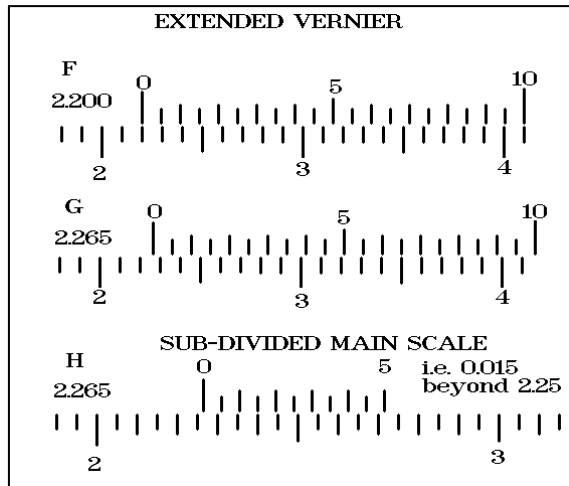
- a) The "0" mark is just beyond a numbered main scale mark, before the first minor tick. Don't lose the zero when adding the vernier value.
- b) The main scale is itself subdivided and the vernier range represents half, typically, of a main scale division. You must decide whether the "0" mark is in the first or second half of the main scale division.
- c) The vernier extends beyond one main scale major division.

At the right are illustrations of regular and 'problem case' vernier readings.

Notice that since the vernier lines are slightly



closer together than the main scale markings, when the best lineup has been selected the slight offset of neighboring vernier lines seem to “lean in” towards the best choice. If you do not see this symmetrical misalignment neighboring the chosen “best” lineup, then the choice was wrong. Try using this test on your choice of best alignment in the examples below.



DISTANCE MEASUREMENTS with Vernier (continued):

Estimated uncertainties for vernier readings are discussed in the answer to self-test question #5, pages 24-25 above.

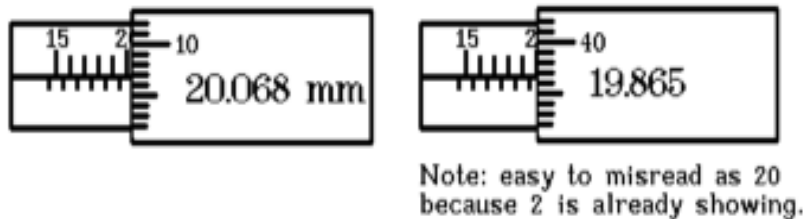
Most vernier calipers have prongs sticking out the back for inside diameter measurements, and some have an extending shaft for depth-of-a-hole measurement.

3. Micrometer: A micrometer is not a C-clamp. It must be used gently, preferably utilizing the slip clutch, to avoid squashing the sample. It is important to check the "zero", i.e. take a reading with no sample and the same pressure as used on the sample.

Since most micrometers use a sub-divided main scale, the key to correctly reading one is to decide which lines on the main scale have been uncovered. This is particularly tricky when one is just short of a main division. The only way to be sure is to: look carefully at the appearance when the "zero" is checked; and observe how far you have to open the jaws, after taking the reading, until the next main scale mark or half-way mark is uncovered.

The direct reading, least count, of a typical micrometer is to $\pm 0.01\text{mm}$, i.e. 10 times more precise than the 10:1 vernier caliper. The estimation place is thus $\pm 0.001\text{ mm}$, though on most samples the readings will not be repeatable to this value.

("cheap" micrometers especially) Sometimes the numbers on the main scale are misleading. For example, enough of the 2, of 20, is uncovered when the actual size is still 19, to fool you - unless you then open the micrometer most of a millimeter until the real 20 appears.



TIMING MEASUREMENTS:

1. The student controlling the timer must be the one observing the action timed. Cooperation in labs is encouraged, but not to the extent of having one student hold the stopwatch and the other tell him or her when to start and stop it. Get the cooperation by taking turns timing successive events or repeated trials.
2. Whenever possible time between similar events. Avoid "sports" timing, between a gunshot and reaching a tape, where reaction time enters only at one end of the interval (since one watches and anticipates the approach to the tape).
3. On repetitive motions (e.g. pendulum swinging) remember the cycle count must start with

zero as the watch is started. A good way to do this is a countdown while watching the motion. For example: 3, 2, 1, 0 (starting timer), 1, 2, etc. Of course, again time between similar events, e.g. top-to-top of a bounce, not release-to-top.

4. *How many swings or 'bounces' should I time?* First you must establish your timing reliability - by successively timing something that is always the same, such as the same mass “bouncing” the same amount while hanging from a spring. [Just because the watch reads to 0.01 seconds doesn't mean your timing is that good. The weak link is in observing the motion and pushing the button.] Practice until your successive trials are consistently within ± 0.2 sec, and usually within ± 0.1 sec of the average. Then if you want, say, 1% accuracy in the period you must time for a total of at least 100 times your timing uncertainty (a minimum of 10 sec. in the example). If you want $\frac{1}{2}$ % accuracy, time for at least 200 times your uncertainty, etc. Note that it is the total time interval measured that counts. It makes no difference whether this was 1 cycle of a slow motion, or 20 of a fast one (unless the timing uncertainty is larger for the slow motion.)

Remember, the gain in accuracy from a long time interval including many cycles of the motion applies only to truly repetitive motion. You can't use it with something whose period is changing, such as a bouncing ball.

5. Do repeated measurements whenever you can. This is not for a statistical reduction in uncertainty, but just to catch a mistake, such as a counting error.
6. (Specific to certain digital stop watches) Take care when reading the stopwatch if the total time is over one minute. The display is confusing.
A reading of 0154 62 is one minute, 54 sec, 62 hundredths
(i.e. 114.62 sec) , NOT 154.62 sec.

MASS MEASUREMENTS

1. Old double-pan balances: Use dynamically, averaging the left and right swing positions. Don't waste time centering the swing without the sample, just note where the no-load balance point is. It is important to place the objects on the center of the pans.
2. Digital balances: Don't drop the sample onto the plate, place it on carefully and well-centered. Check that the expected units are displayed; most digital balances are multi-scale.

MULTIMETERS (Volts, Amps, Ohms)

To use a multi-function meter you have to make two choices: function selection and range.

The **function selector** tells the meter whether you are measuring voltage, current, or resistance. This should always be set before connecting the meter to whatever is to be measured.

The **range** should be chosen to be larger than the maximum reading expected. It, too, should be chosen before connecting the meter. The range and function selection together determine the “units” of the measurement.

On **analog meters** the range switch tells which of the several printed scales to read. There are several scales, for example: 0 to 5, 0 to 10, and 0 to 25. One reads the scale that ends with the range selected. For example, if the needle position is half of full scale on a range of 10, one reads the scale ending with 10 and gets a value of 5. The same needle position on a range of 25 would be the result of an input of 12.5. If the range is set on 100, one reads the “10” scale and mentally moves the decimal to make the reading 50. And a midway position on the 2.5 range would be read as 1.25 on the “25” scale. It is up to the user to read the correct scale and place the decimal point appropriately. Generally, for greatest accuracy, one should choose a range that puts the needle position in the upper half of the selected scale. If the range switch is set too low the needle goes off scale, potentially damaging the meter mechanically if it hits the end too hard.

On **digital meters** the range selection is one digit more than the maximum value readable on that range. For example, on a range of 20 volts, the maximum readable voltage is 19.999V. If the value being measured is larger than the range, an error is indicated by a flashing display, or an “e”, or, most often, by a reading of “1.____”, depending on the meter maker. The meter places the decimal, and the user needs only to note whether the range label is, say, millivolts, volts, or kilovolts.

Unless the multimeter is a “permanent” part of the setup, it should always be disconnected and left on “OFF” or a high “Volts” range to protect it against the next user’s thoughtless connection before selections.

OVERALL DATA REMINDER

Always record the value, units, and estimated error. Then look at what you've written down and ask yourself, "Is this reasonable?" It's easy to correct a data error "on the spot"; it's very hard to correct it a week later when writing the report!

MORE SELF-TEST QUESTIONS

1. The motor displacement on a typical 1960's "muscle car" was 440 cubic inches. What is this in the more modern statement of liters?
2. You measured the "small angle" period of a pendulum twice. At the count of ten swings you had 13.52 sec; for the count of 20 you had 28.48 sec. What is the period.
3. You've measured the volume of a cylinder with an axial hole two ways: by measuring its dimensions, and by measuring the bouyant force on it when immersed in water and using Archimede's principle. Are the two results "consistent"? Here is your data.
 - outside diameter 3.816 ± 0.004 cm
 - hole diameter 0.42 ± 0.02 cm
 - cylinder length 3.902 ± 0.004 cm
 - scale reading dry 356.2 ± 0.2 grams
 - scale rdg. submerged 312.5 ± 0.3 gma
 - density of water 1.00 gms/cm³ (assumed)
4. Theory says that the period of oscillation T of a mass on a spring is $2\pi(M_{TOT}/k)^{1/2}$, where M_{TOT} is the total mass (i.e. added mass M plus effective mass m of the spring), and k is the spring "stiffness."

Plot the following data to find m and k

M (added)	T
50 grams	0.548 sec
150	.775
250	1.054
450	1.223
1950	2.450
2950	3.000

MORE SELF-TEST ANSWERS:

1. $440 \text{ in}^3 \left(\frac{2.54 \text{ cm}}{1 \text{ in}} \right)^3 \left(\frac{1 \text{ liter}}{1000 \text{ cm}^3} \right) = 7.21 \text{ liter}$

2. $13.52/10 = 1.352 \pm .02$; $28.48/20 = 1.424 \pm .01$

[\pm values based on generous 0.2 sec timing uncertainty estimate]

Clearly inconsistent. You probably started counting with 1 instead of 0.

This assumption would give you $13.52/9 = 1.502$ and $28.48/19 = 1.499$.

These are consistent, but if this were a real situation you better go back and re-time to be sure.

3. Volume by dimensions $44.09 \pm .11 \text{ cm}^3$

important intermediate steps:

$$\Delta(D^2) = 2 D \Delta D = 0.0305; \text{ similarly } \Delta(d^2) = 0.0168$$

$$\Delta V/V = \text{sqrt}\{(\Delta H/H)^2 + [\Delta(D^2 - d^2)]^2\} = .0026$$

Volume by immersion: 43.6 ± 0.2

difference of 0.5 cm^3 is greater than $\text{sqrt}(.11^2 + .2^2) = 0.3$

therefore not “consistent”, though close

(better check on that assumed water density)

4. Clearly the best way is to square both sides of the theory equation and then recognize that since $T^2 = (4\pi^2/k)(M+m)$, plotting T^2 vs M will give a straight line of slope $4\pi^2/k$ and x-intercept (at $T=0$) of $-m$.

First, note that the data point for $M = 250$ is significantly off the line, and should be discarded. Then, planning ahead for the negative mass intercept, you get $m = 50 \text{ gm}$ and $k = 1.316 \times 10^4 \text{ dynes/cm}$.

If you do this on a spreadsheet, you will take the numbers off the trend line equation. But for the satisfaction of seeing the intercept plotted you can use the trend line’s “forecast” option, or simply add a negative “M” data point (e.g. -100) with empty T value and force the computer to extrapolate its drawn line.

There are two other interesting options if doing the plot on a spreadsheet. Both require adding a column to allow adding a constant to the M data, for example ($M_{\text{COLUMN}} + \$E\1) to use as the “x” variable. Then you can do either a power law fit, or a log-log plot, adjusting the value in $\$E\1 for a best trendline fit as measured by the R^2 fit parameter. (Incidentally, you will want to “format data labels” for several decimal places.) Then, by trial and error, you will find m must be between about 46 and 54 grams for

$R^2 \cong 1$, and you have the satisfaction of “discovering” that the power is close to the theoretical 0.5.

One other “discovery” you can make on the spreadsheet is to let it fit a polynomial to the T vs M data. By the time you get to a 4th power polynomial, the fit as measured by R^2 is excellent. But look at that fit line. Obvious nonsense!

A FINAL “ANALYSIS” CONCLUSION.

ALWAYS:

Check the answer for appropriate units.

Verify that the \pm estimate is consistent with the “significant figures.”

Verify that appropriate theory has been used to guide analysis.

USUALLY:

Compare your answer to “accepted” values, or to other measurements.

Attempt to account for a difference in the above, either through carefully estimated uncertainty in the measurements or by analysis of possible systematic errors.

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