Error Analysis

When we measure physical quantities we are always faced with the limitations of the instruments. How accurately can we make a measurement? The purpose of error analysis, probably better called uncertainty analysis, is to establish reasonable limits to the values we report. A detailed account of error analysis can be found in Professor Carr's supplementary notes on errors and graphing. In this short discussion we will only cover the application of error propagation to a few cases to illustrate the procedure.

Propagation of Errors

Suppose we can directly measure two quantities, x and y. The instruments allow us to determine these values only up to a certain level of accuracy. You, as the experimenter, are the judge of this level of accuracy. There is no hard and clear rule for determining the uncertainty of the instrument. Your judgment of the uncertainty that is to be attached to the values is what matters since your name will be on the final result. The reader of your report will rely on your good sense in assessing the uncertainty in the reported results. From your measured values of x and y you are going to determine a derived experimental result, call it f, namely

$$f = f(x, y) \tag{1}$$

Your responsibility as the experimenter is to report not only the best value of f but also the uncertainty you attach to the result, that is, $f \pm \Delta f$. Your instruments do not give you the value of Δf directly. Rather, you must deduce that uncertainty from the uncertainties in your two measured values, $\Delta x, \Delta y$. The steps you take to get at Δf from $\Delta x, \Delta y$ constitute the propagation of error analysis. We will assume that the uncertainties in x and y are independent from each other. That is, you can measure the quantity x independently from whatever value y might have, for example. We will consider two cases of combining x and y to get f.

Case 1: $f = ax + by$

Consider the case where a, b are constants for which there are no uncertainties. Your measured quatities are $x \pm \Delta x$ and $y \pm \Delta y$. Since x, y are independent we use a rule called combining the errors in quadrature. We determine the uncertainty in f by

$$(\Delta f)^2 = a^2 (\Delta x)^2 + b^2 (\Delta y)^2 \tag{2}$$

Since $\Delta f = \sqrt{(\Delta f)^2}$ your final answer is $f \pm \Delta f$.

Case 2: $f = axy$ **or** $f = ax/y$ Again a is a constant for which there is no uncertainty. In this case there is a simpler formulation by expressing the fractional uncertainties. Again we can combine the uncertainties in quadrature.

$$(\Delta f/f)^2 = (\Delta x/x)^2 + (\Delta y/y)^2 \tag{3}$$

Since $\Delta f = f\sqrt{(\Delta f/f)^2}$ your final answer is $f \pm \Delta f$.

Example: Obtaining the volume of a cylinder with an axial hole in it

We use the calipers to obtain values for the outer diameter D and the inner diameter d and the length L. We judge that we can determine distance with the calipers to within $0.002cm$. Suppose now $D = 3.800 \pm 0.002cm$, $d = 0.445 \pm 0.002cm$ and $L = 3.790 \pm 0.002cm$. The derived experimental quantity is the volume V,

$$V = (\pi/4)L(D^2 - d^2) = abc \tag{4}$$

Where $a = (\pi/4)$, $b = L$ and $c = (D^2 - d^2)$. V is a product of three terms. Since the term involving a is a constatnt there is no uncertainty associated with it. The terms involving b, c do have uncertainties. The expression for the fractional uncertainties is then

$$(\Delta V/V)^2 = (\Delta b/b)^2 + (\Delta c/c)^2 = (\Delta L/L)^2 + (\Delta(D^2 - d^2)/(D^2 - d^2))^2 \tag{5}$$

$$(\Delta L/L)^2 = (0.002/3.790)^2 = 2.78 \times 10^{-7} \tag{6}$$

The term for $\Delta(D^2 - d^2)$ is in the form of case 1. We must obtain the uncertainties for the terms D^2, d^2 separately. For example, the uncertainty in D^2 is of the form of case 2 where $x = D, y = D$, hence

$$(\Delta DD/DD)^2 = (\Delta D/D)^2 + (\Delta D/D)^2 = 2(\Delta D/D)^2 \tag{7}$$

$$\Delta D^2/D^2 = \sqrt{2}\Delta D/D \tag{8}$$

$$\Delta D^2 = \sqrt{2}D\Delta D = \sqrt{2}(3.80) * 0.002cm^2 = 1.07 \times 10^{-2}cm^2 \tag{9}$$

So likewise for Δd^2

$$\Delta d^2 = \sqrt{2}d\Delta d = \sqrt{2} * 0.445 * 0.002 = 1.3 \times 10^{-3}cm^2 \tag{10}$$

$$(\Delta(D^2 - d^2))^2 = (1.07 \times 10^{-2}cm^2)^2 + (0.13 \times 10^{-2}cm^2)^2 = 1.16 \times 10^{-4}cm^4 \tag{11}$$

Then

$$(\Delta(D^2 - d^2)/(D^2 - d^2))^2 = (1.16 \times 10^{-4}cm^4/(14.24cm^2)^2 = 5.72 \times 10^{-7} \tag{12}$$

Substituting these fractional uncertainties gives

$$(\Delta V/V)^2 = (2.78 + 5.72) \times 10^{-7} \tag{13}$$

We evaluate the volume, $V = (\pi/4) * (3.790) * 14.24cm^3 = 42.39cm^3$, $\Delta V/V = 9.2 \times 10^{-4}$, $\Delta V = 42.39 * 9.2 \times 10^{-4}$ and we report $V = 42.39 \pm 0.04cm^3$.

General formula for error propagation by quadrature

The formulas we used in eqn 2 and eqn 3 can be derived from the general expression, eqn 14, for combining independent errors using quadrature. Given

a function $f(x_1, x_2, ...x_n)$ of n independent variables we propagate the errors in the $x_1, x_2, ...x_n$ to obtain the error in f.

$$(\Delta f)^2 = \sum_{i=1}^{i=n} (\frac{\partial f}{\partial x_i})^2 (\Delta x_i)^2. \tag{14}$$

Consider the example we just used, $V(L, D, d) = (\pi/4)L(D^2 - d^2)$ and evaluate the partial derivatives.

$$\frac{\partial V}{\partial L} = (\pi/4)(D^2 - d^2). \tag{15}$$

$$\frac{\partial V}{\partial D} = (\pi/4)L(2D) \tag{16}$$

$$\frac{\partial V}{\partial d} = -(\pi/4)L(2d) \tag{17}$$

Evaluating the expression in eqn 18

$$(\Delta V)^2 = ((\pi/4)(D^2 - d^2)\Delta L)^2 + ((\pi/4)L(2D)\Delta D)^2 + (-(\pi/4)L(2d)\Delta d)^2 \tag{18}$$

Gives $\Delta V = 0.046 cm^3$, which is the same as our earlier result except for rounding off errors.

Comparing experimental results to each other and to theory

Many times the same physical quantity is measured by several groups. It is highly unlikely that all these results will be exactly the same given experimental uncertainties. How then does one compare the different values to each other or to a theoretical expectation? The comparison can only be done if the uncertainties have been reported. Suppose the same cylinder we have been discussing has been measured by five groups with values in cm^3 $V_0 = 42.39, V_1 = 42.32, V_2 = 42.26, V_3 = 42.19, V_4 = 42.10$. Each group claims to have the same uncertainty, $\Delta V = 0.05 cm^3$. Is there a significant difference between the different measurements? We can compare group 0 to the other groups, for example. Call the difference between the absolute value between group 0 and group 1 δ_{01}.

$$\delta_{01} = |42.39 - 42.32| = 0.07 cm^3 \tag{19}$$

The uncertainty in this difference is given by the formula in eqn 2

$$(\Delta(diff))^2 = (\Delta V_0)^2 + (\Delta V_1)^2 = (0.05 cm^3)^2 + (0.05 cm^3)^2 \tag{20}$$

$$\Delta(diff) = \sqrt{2} * 0.05 cm^3 = 0.07 cm^3 \tag{21}$$

The difference between groups 0 and 1 is said to be within experimental error since in this case the uncertainty in the difference is about the same size as the difference, $\delta_{01} \approx \Delta(diff)$. The difference between group 0 and group

2 is $\delta_{02} = 0.13cm^3$ but the uncertainty in the difference is the same as before since each group claims the same uncertainty in their measurements of $0.05cm^3$. Here $0.13cm^3 > 0.07cm^3$, but in this case also we conclude that there is no significant difference between groups 0 and 2 because it is still statistically possible that this size difference is compatible with the measurement uncertainties. A comparison between group 0 and group 4 shows even a larger difference, $\delta_{04} = |42.39 - 42.10| = 0.29cm^3$. Here δ_{04} is 4 times bigger than the uncertainty in the difference. In this case the conventional assessment in error analysis is that there is a significant difference. Either one or the other or both groups have made an error in the measurement. The conventional assessment in error analysis often uses the following scheme to determine the agreement between values i and j :

If $0 \le \delta_{ij} \le 2 * \Delta(diff_{ij})$ then there is no significant difference.

If $\delta_{ij} \ge 3 * \Delta(diff_{ij})$ then there is a significant difference.

If $2 * \Delta(diff_{ij}) \le \delta_{ij} \le 3 * \Delta(diff_{ij})$ the results are fuzzy and it is wise to redo the measurement and try to improve on the accuracy so that a definite statement on the agreement or disagreement can be made.

We can also compare experiment to theory using this formulaic prescription. Stating a comparison in this fashion will enable you to communicate your results in a way that other experimenters can understand.