Kinematics

All kinematics problems rely on the conservation of energy and momentum. You should not worry about the nature of the forces involved in the reactions!

Non Relativistic Kinematics

Lab to Center of Mass transformation

Consider the reaction

$$m_a + m_b \to m_c + m_d. \tag{1}$$

The lab coordinates of particles a and b are related to the center of mass coordinates, see Figure 1, by $\vec{R}a = \vec{R} + \vec{ra}$ and $\vec{R}b = \vec{R} + \vec{rb}$. Hence $m_a\vec{R}a = m_a\vec{R} + m_a\vec{ra}$, and, $m_b\vec{R}b = m_b\vec{R} + m_b\vec{rb}$. Take the time derivative and add these two equations together.

$$m_a\vec{V}a + m_b\vec{V}b = m_a\vec{V} + m_a\vec{va} + m_b\vec{V} + m_b\vec{vb}. \tag{2}$$

In the center of mass the total momentum is zero, so

$$m_a\vec{V}a + m_b\vec{V}b = (m_a + m_b)\vec{V}. \tag{3}$$

First evaluate this in the center of mass(CM). Let $\vec{pa}, \vec{pb}, \vec{pc}, \vec{pd}$ be the CM momenta so that the magnitudes of $\vec{pa}, \vec{pb}$ are equal and set that to pi. In the final state set the magnitudes of $\vec{pc}, \vec{pd}$ to pf. The conservation of momentum is satisfied in the CM in the initial and final states because the sum of the momenta is zero both before and after. The conservation of energy requires

$$t_a + t_b + m_a + m_b = t_c + t_d + m_c + m_d \tag{4}$$

$$t_a + t_b = -Q + t_c + t_d \tag{5}$$

$$Q = m_a + m_b - m_c - m_d \tag{6}$$

or expressing the kinetic energies non relativistically

$$pi^2/(2m_a) + pi^2/(2m_b) = -Q + pf^2/(2m_c) + pf^2/(2m_d). \tag{7}$$

Now consider the relationship between the laboratory momenta of a and b and the CM momenta of a and b. $Pa = m_a\vec{V} + \vec{pa}$, $Pb = m_b\vec{V} + \vec{pb}$ or in terms of the CM momentum $P = (m_a + m_b)\vec{V} = M\vec{V}$.

$Pa = (m_a/M)\vec{P} + \vec{pa}$, $Pb = (m_b/M)\vec{P} + \vec{pb}$.

Hence the momentum $|pa| = |pb| = |pi|$ in the CM is given by

$$pi^2 = pa^2 = (m_b/M)^2 Pa^2 + (m_a/M)^2 Pb^2 - (2m_a m_b/M^2)\vec{Pa} \cdot \vec{Pb}. \tag{8}$$

From this relation we can link the kinetic energies in the lab to the kinetic energies in the CM because $T_a = Pa^2/(2m_a)$. If we specialize to the case that b is stationary in the lab then $P_b = 0$ and then in the CM

$$pi^2/(2m_a) + pi^2/(2m_b) = (m_b/M)^2 Pa^2(1/m_a + 1/m_b)/2 = -Q + t_a + t_b.$$

$$2m_a(m_b/M)^2 T_a(M/(m_a m_b)/2 = -Q + t_a + t_b.$$

$$T_a = (m_a + m_b)/m_b(-Q + t_a + t_b). \tag{9}$$

If Q is negative then we can have $t_a = t_b = 0$ and still satisfy $pf = p_a = p_b = 0$. The minimum lab energy, non relativistically, to cause the reaction occurs when the kinetic energies of c and d are zero in the CM.

$$T_a^{min} = -Q(1 + ma/mb) \tag{10}$$

In this case since $pf = 0$ the particles c and d do not separate from each other in the CM and likewise they stay stuck together in the lab.

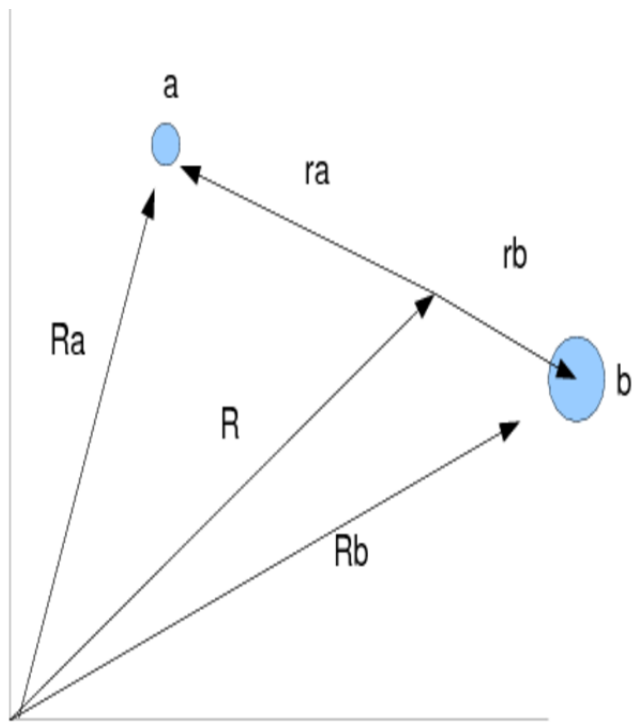

Figure 1: Two particles of masses m_a and m_b are at displacements $\vec{Ra}$ and $\vec{Rb}$ from the lab origin of coordinates. The center of mass is $\vec{R}$. The locations of a and b with respect to the center of mass are $\vec{ra}$ and $\vec{rb}$.

Particle momenta versus scattering angle, non relativistic treatment

Continuing with the assumption that particle b is stationary and that particle c is detected at an angle θ_c in the lab with respect to the initial momentum $\vec{P}a$ we want to determine the momentum of particle c. We use the non relativistic expression for kinetic energy in equation 4

$$T_a + Q = Pc^2/(2m_c) + Pd^2/(2m_d) \tag{11}$$

Conservation of momentum requires that

$$\vec{P}d = \vec{P}a - \vec{P}c \tag{12}$$

Substitute equation 12 into the energy equation 11.

$$T_a + Q = Pc^2/(2m_c) + (Pa^2 + Pc^2 - 2PaPccos(\theta_c))/(2m_d) \tag{13}$$

Equation 13 is quadratic in the lab momentum Pc. We solve this equation and obtain:

$$Pc = \beta \pm \sqrt{\alpha + \beta^2} \tag{14}$$

Where $\beta = 2m_cPacos(\theta_c)/(m_c + m_d)$

and

$\alpha = 2(Ta(1 - m_a/m_d) + Q)m_cm_d/(m_c + m_d)$.

Conservation of momentum and energy now allows us to get $\vec{P}d$ as well.

Relativistic Kinematics

Lab to Center of Mass transformation

We again consider the reaction in equation 1. In the case of relativistic kinematics we have a different expression for the kinetic energy term, i.e., the correct expression. We still can write

$E = T + m$, but also note that

$$E = \sqrt{p^2 + m^2} \tag{15}$$

In this case we form the 4-momentum, $p^\mu = (E, \vec{p})$ and use the Lorentz invariant contraction

$$S = p^\mu p_\mu = E^2 - p^2 \tag{16}$$

which gives the same result in the lab or in the center of mass. The total momentum in the center of mass is, by definition, zero. Hence

$$S = E_{lab}^2 - P_{lab}^2 = E_{cm}^2 \tag{17}$$

For particle b stationary we have $E_{lab} = m_a + m_b + Ta$, and $\vec{P_{lab}} = \vec{P}a$. In the center of mass we have

$$E_{cm} = ec + ed = \sqrt{m_c^2 + pc^2} + \sqrt{m_d^2 + pd^2}$$

However, $\vec{pc} + \vec{pd} = 0$. Call the magnitude of pc and pd pf so then

$$E_{cm} = ec + ed = \sqrt{m_c^2 + pf^2} + \sqrt{m_d^2 + pf^2}$$

Since we know S in the lab from equation 17 we have

$$S = m_c^2 + m_d^2 + 2pf^2 + 2\sqrt{m_c^2 + pf^2}\sqrt{m_d^2 + pf^2} \tag{18}$$

Let $\alpha = S - (m_c^2 + m_d^2)$ then

$$\alpha - 2pf^2 = 2\sqrt{m_c^2 + pf^2}\sqrt{m_d^2 + pf^2} \tag{19}$$

We square both sides of this equation to get

$$\alpha^2 + 4pf^4 - 4\alpha pf^2 = 4(m_c^2 + pf^2)(m_d^2 + pf^2) \tag{20}$$

The terms $4pf^4$ cancel and we are again left with a quadratic equation in pf^2.

$$\alpha^2 - 4m_c^2 m_d^2 = 4pf^2(mc^2 + md^2 + \alpha) \tag{21}$$

The magnitude of the momentum of particles b and d in the center of mass is

$$pf^2 = (\alpha^2 - 4m_c^2 m_d^2)/(4(m_c^2 + m_d^2 + \alpha)) \tag{22}$$

When $pf^2 = 0$ this is the case of the minimum energy required in the center of mass to produce the reaction so $\alpha = 2m_c m_d$. Using $Ea = Ta + m_a$ and evaluating α

$$\alpha = (Ea+m_b)^2 - Pa^2 - (m_c^2+m_d^2) = Ea^2 + 2Eam_b + m_b^2 - Pa^2 - (m_c^2+m_d^2) \tag{23}$$

$$\alpha = Pa^2 + m_a^2 + 2(Ta + m_a)m_b + m_b^2 - Pa^2 - (m_c^2 + m_d^2) \tag{24}$$

$$\alpha = m_a^2 + m_b^2 + 2m_a m_b + 2Tam_b - (m_c^2 + m_d^2) \tag{25}$$

$$\alpha = (m_a + m_b)^2 + 2Tam_b - (mc^2 + m_d^2) \tag{26}$$

So then substituing for α

$$(m_a + m_b)^2 + 2Tam_b - (m_c^2 + m_d^2) = 2m_c m_d \tag{27}$$

$$2Tam_b = (m_c + m_d)^2 - (m_a + m_b)^2 \tag{28}$$

Using the fact that $(m_c + m_d) = (m_a + m_b) - Q$ and solving for Ta

$$Ta^{min} = -Q(m_a + m_b)/m_b + Q^2/(2m_b) \tag{29}$$

Comparing the relativisitic value for Ta^{min}, equation 29, with the nonrelativistic value, equation 10, we see the correction to the nonrelativistic value is, $Q^2/(2m_b)$.

Particle momenta versus scattering angle, relativistic treatment

We use the conservation laws, $\vec{Pa} = \vec{Pc} + \vec{Pd}$ and $Ea + Eb = Ec + Ed$. So $\vec{Pc} = \vec{Pa} - \vec{Pd}$ and with $E_i = \sqrt{Pa^2 + m_a^2} + m_b = \sqrt{Pc^2 + m_c^2} + \sqrt{Pd^2 + m_d^2}$ we can make the substitution and rearrange the equation

$$E_i - \sqrt{Pc^2 + m_c^2} = \sqrt{Pd^2 + m_d^2} = \sqrt{m_d^2 + Pa^2 + Pc^2 - 2PaPccos(\theta_c)}.$$

Square both sides and make the substitution $\alpha = E_i^2 + m_c^2 - m_d^2 - Pa^2$

$$\alpha + 2PaPccos(\theta_c) = 2E_i\sqrt{m_c^2 + Pc^2} \tag{30}$$

Squaring again we solve the quadratic equation to obtain Pc as a function of scattering angle θ_c.

$$Pc = \beta \pm \sqrt{\gamma + \beta^2} \tag{31}$$

With $\beta = \alpha Pacos(\theta_c)/(2(E_i^2 - Pa^2cos^2(\theta_c)))$
and $\gamma = (\alpha^2 - 4E_i^2m_c^2)/(4(E_i^2 - Pa^2cos^2(\theta_c)))$
and Pc must be greater or equal to zero.

Using the Lorentz transformation to the Center of Mass to obtain the Q value

Suppose the momentum of particle a, $\vec{P_a} = P_a\hat{z}$, is along the z axis. The four momentum of particles a and b can be transformed to another inertial reference frame, O' moving along the z axis with velocity β, by the Lorentz transformation. Here $P_a^\mu = (E_a, \vec{P_a})$, $\vec{P_a} = (0, 0, P_a)$ so

$$E'_a = \gamma(E_a - \beta P_{az}) \tag{32}$$

$$p'_{az} = \gamma(P_{az} - \beta E_a) \tag{33}$$

$$p'_{ax} = 0 \tag{34}$$

$$p'_{ay} = 0 \tag{35}$$

Here $\gamma = 1/\sqrt{1-\beta^2}$. If particle b is stationary in the lab then $\vec{P_b} = (0, 0, 0)$ so then

$$E'_b = \gamma(E_b - \beta P_{bz}) = \gamma E_b \tag{36}$$

$$p'_{bz} = \gamma(P_{bz} - \beta E_b) = -\gamma\beta E_b \tag{37}$$

If we want the frame O' to be the Center of Mass we recall that in the Center of Mass the total momentum is zero.

$$p'_{az} + p'_{bz} = 0 = \gamma(P_{az} - \beta E_a - \beta E_b) \tag{38}$$

This equation allows us to get the Center of Mass velocity β

$$\beta = P_{az}/(E_a + E_b) \tag{39}$$

If we have the Center of Mass velocity and in the lab we measure a particle c with momentum $\vec{P_c} = (P_{cx}, P_{cy}, P_{cz})$ then we can obtain the momentum of particle c in the Center of Mass using the Lorentz transformation, equation 33.

$$p_c'^2 = \gamma^2(P_{cz} - \beta E_c)^2 + P_{cx}^2 + P_{cy}^2 \tag{40}$$

In the Center of Mass $p_d'^2 = p_c'^2$ and from the conservation of energy we have a relationship for m_d.

$$E_i' = \sqrt{p_c'^2 + m_c^2} + \sqrt{p_d'^2 + m_d^2} \tag{41}$$

Where

$$E_i'^2 = (E_a + E_b)^2 - P_a^2 = s = p_\mu p^\mu \tag{42}$$

We have another way of writing the equation for m_d using

$$E_c' = \gamma(E_c - \beta P_{cz}) \tag{43}$$

Then

$$m_d^2 = E_i'^2 - 2E_i'E_c' + m_c^2 \tag{44}$$

The center of mass velocity for a multi particle system

Suppose we have N particles in a system. The total laboratory momentum of these particles is given by

$$\vec{P} = \sum_{i=1}^{N} \vec{P_i} \tag{45}$$

In Figure 2 a particular vector $\vec{P_i}$ makes an angle θ with respect to the total momentum. The plane in this figure is perpendicular to $\vec{P}$. The component of $\vec{P_i}$ which is in this plane is $\vec{P_i}sin(\theta_i)$ and the component of $\vec{P_i}$ along $\vec{P}$ is $\vec{P_i}cos(\theta_i)$ thus

$$\vec{P} \cdot \sum \vec{P_i} = P^2 = P \sum P_i cos(\theta_i) \tag{46}$$

so $P = \sum_{i=1}^{N} P_i cos(\theta_i)$. We also have

$$\vec{P} \times \vec{P_i} = PP_i sin(\theta_i)\hat{n} \tag{47}$$

and we add all the perpendicular components together to get

$$\sum \vec{P} \times \vec{P_i} = \vec{P} \times \vec{P} = \vec{P} \sum P_i sin(\theta_i)\hat{n} \tag{48}$$

So the sum of the perpendicular components is zero.

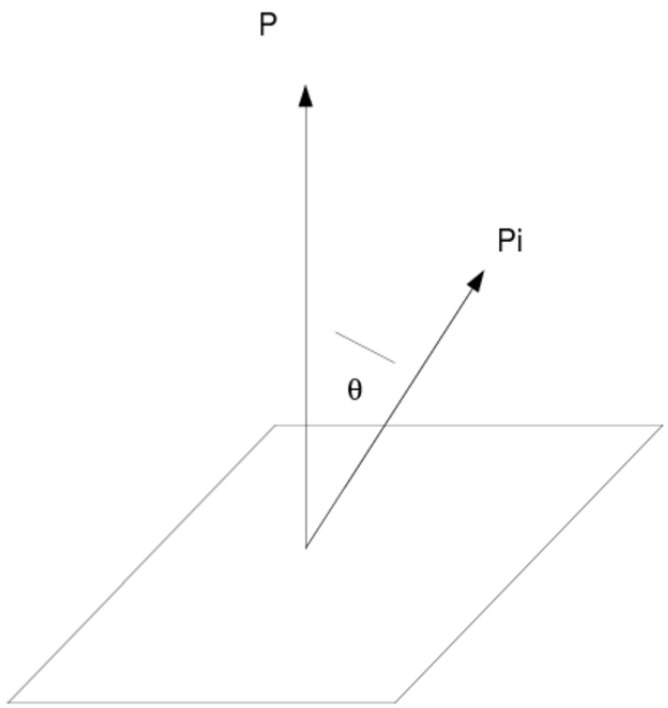

Figure 2: The total momentum $\vec{P} = \sum \vec{P_i}$. A particular momentum $\vec{P_i}$ makes an angle of θ with respect to $\vec{P}$. The plane is perpendicular to $\vec{P}$. The components $\sum P_i \vec{sin}(\theta_i)$ add to zero.